

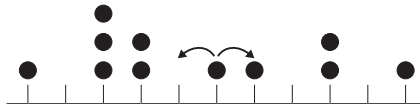
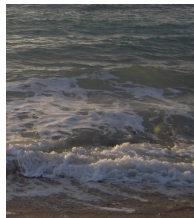
Stochastic approach to a space-time scaling limit for Hamiltonian systems

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From microscopic world to macroscopic world



Motivation of our study:

“Explain **macroscopic phenomena** from **microscopic dynamics**”

In particular, a derivation of **diffusion phenomena** is the main interest of this talk

Micro and Macro

	Micro	Macro
Physical quantity	position or velocity of each molecular	density, temperature, pressure <small>etc</small>
Degree of freedom	enormous	a few
Time evolution	complicated interaction of elements	PDE

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Physical quantity	position or velocity of each molecular	density, temperature, pressure <small>etc</small>
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Time evolution	complicated interaction of elements	PDE

- Each microscopic quantity has not (almost) any information for the macroscopic state
- Statistics (average, fluctuation etc) of microscopic quantities determines the macroscopic state
- These properties have good compatibility with stochastic analysis

How to connect microscopic dynamics to macroscopic diffusion?

A typical procedure is the following:

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- Rescale the model **in space and time** with scaling parameter N and take **scaling limit** $N \rightarrow \infty$

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Deterministic or Stochastic?

If the microscopic dynamics is a **deterministic** classical dynamical system, then ...

- The microscopic dynamics is time reversible, but the diffusion equation is not!
- As $N \sim 10^{23}$ or more, it is impossible to solve or even simulate the dynamics
- It is very difficult to justify that we can forget about “microscopic quantities” in the limit

We often consider a **stochastic process** as a microscopic model and study **scaling limits** for the process

- Hydrodynamic limit
- Equilibrium fluctuation

Hydrodynamic limit and Equilibrium fluctuation

Hydrodynamic limit is...

- rigorous method to derive **deterministic** macroscopic PDEs
- law of large numbers (LLN)

Equilibrium fluctuation is...

- rigorous method to derive **stochastic** macroscopic PDEs
- central limit theorem (CLT)

Example. Symmetric Simple Exclusion Process (SSEP)

- Continuous time symmetric random walks with hard core interaction



HDL limit → Properly scaled **density of particles ρ** and its fluctuation **Y** evolve according to

$$\partial_t \rho = \frac{1}{2} \Delta \rho, \quad dY_t = \frac{1}{2} \Delta Y_t dt + \sqrt{\rho_t(1 - \rho_t)} \nabla dB_t$$

① Typical stochastic models

Symmetric Simple Exclusion Process (SSEP)

Totally Asymmetric Simple Exclusion Process (TASEP)

② Hamiltonian dynamics + stochastic noise

③ Two-step approach

Stochastic energy exchange model

Energy conserving stochastic Ginzburg-Landau model

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State space of SSEP and TASEP

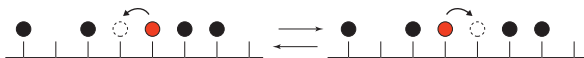
\mathbb{Z}^d : d -dimensional discrete lattice

$\chi^d := \{0, 1\}^{\mathbb{Z}^d}$: state space, $\eta = (\eta_x)_{x \in \mathbb{Z}^d}$: element of χ^d

- $\eta_x = 1$: there is a particle at x
- $\eta_x = 0$: there is no particle at x



Symmetric Simple Exclusion Processes (SSEP)



- Continuous time Markov process on χ^d
- jump to one of the neighboring sites with probability $\frac{1}{2d}$
- jump rate (the inverse of the expectation value of **random waiting time**) is a **constant 1**
- exclusion rule

$\{\eta^t\}_{t \geq 0}$: Markov process on χ^d with generator

$$(Lf)(\eta) = \frac{1}{2d} \sum_{|x-y|=1} \mathbf{1}_{\{(\eta_x, \eta_y) = (1, 0)\}} (f(\eta^{x \rightarrow y}) - f(\eta))$$

$$Lf = \frac{d}{dt} P_t f|_{t=0}, \quad P_t: \text{Markov semigroup}, \quad P_t f(\eta) = E_\eta[f(\eta^t)]$$

- The number of particles is a unique conserved quantity
 \Rightarrow The density of particles characterizes the equilibrium (macroscopic) states ($\{\nu_\rho\}_{\rho \in [0,1]}$: Bernoulli product measures)
 \Rightarrow Derive an evolution equation of the density of particles

For simplicity, we consider the discrete torus $\mathbb{T}_N^d = \{1, 2, \dots, N\}^d$ instead of \mathbb{Z}^d from now on.

Denote by π_t^N the scaled empirical measure under **diffusive scaling**:

$$\pi_t^N(du) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_x(N^2 t) \delta_{\frac{x}{N}}(du) \in \mathcal{M}(\mathbb{T}^d = [0, 1]^d)$$

($\mathcal{M}(\mathbb{T}^d = [0, 1]^d)$: set of measures on \mathbb{T}^d)

Hydrodynamic limit for SSEP

Theorem (De Masi, et al. 1984)

Assume

$$\pi_0^N(du) \rightarrow \pi_0(du) = \rho_0(u)du \quad N \rightarrow \infty \quad \text{in prob}$$

with some measurable function $\rho_0 : \mathbb{T}^d \rightarrow [0, 1]$.

Then, $\forall t > 0$,

$$\pi_t^N(du) \rightarrow \pi_t(du) = \rho(t, u)du \quad N \rightarrow \infty \quad \text{in prob}$$

where $\rho(t, u)$ is the unique solution of the heat equation:

$$\begin{cases} \partial_t \rho(t, u) = \frac{1}{2d} \Delta \rho(t, u) \\ \rho(0, \cdot) = \rho_0(\cdot) \end{cases}$$

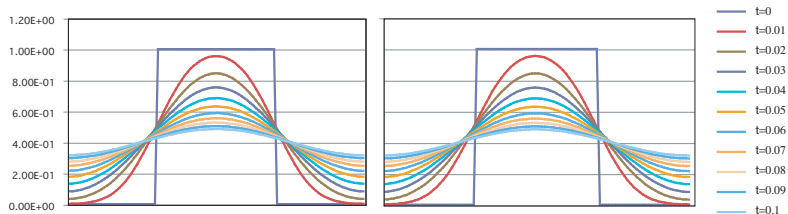
How to guess the limiting equation

Assume $\rho(t, u) \sim \mathbb{E}[\eta_x^s]$ for $u = \frac{x}{N}$ and $t = \frac{s}{N^2}$

$$\begin{aligned}\partial_t \rho(t, u) &\sim \mathbb{E}[N^2 L \eta_x^s] \\ &= \frac{N^2}{2d} \sum_{i=1}^d \mathbb{E}[\eta_{x+e_i}^s (1 - \eta_x^s) + \eta_{x-e_i}^s (1 - \eta_x^s) \\ &\quad - \eta_x^s (1 - \eta_{x+e_i}^s) - \eta_x^s (1 - \eta_{x-e_i}^s)] = \frac{N^2}{2d} \sum_{i=1}^d \mathbb{E}[\eta_{x+e_i}^s - 2\eta_x^s + \eta_{x-e_i}^s] \\ &\sim \frac{1}{2d} \Delta^N \rho(t, u) \xrightarrow{N \rightarrow \infty} \frac{1}{2d} \Delta \rho(t, u)\end{aligned}$$

$$\Delta^N H(u) = \sum_{i=1}^d N^2 \left\{ H\left(u + \frac{e_i}{N}\right) - 2H(u) + H\left(u - \frac{e_i}{N}\right) \right\}$$

Numerical simulation



*exact solution of heat eq.

*numerical simulation of SSEP

$N = 100$

averaged density of 5,000,000 paths

Totally Asymmetric Simple Exclusion Process (TASEP)

\mathbb{Z} : 1-dimensional discrete lattice

$\chi := \{0, 1\}^{\mathbb{Z}}$: state space $\eta = (\eta_x)_{x \in \mathbb{Z}}$: element of χ



- jump **only to right**
- jump rate is a constant 1
- exclusion rule

$$(Lf)(\eta) = \sum_{x \in \mathbb{Z}} 1_{\{(\eta_x, \eta_{x+1}) = (1, 0)\}} (f(\eta^{x \rightarrow x+1}) - f(\eta))$$

Hydrodynamic limit for TASEP

Denote by π_t^N the scaled empirical measure under **hyperbolic scaling**:

$$\pi_t^N(du) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta_x(Nt) \delta_{\frac{x}{N}}(du) \in \mathcal{M}(\mathbb{R})$$

Theorem (Rezakhanlou, 1991)

Assume some conditions for π_0^N . Then, $\forall t > 0$,

$$\pi_t^N(du) \rightarrow \pi_t(du) = \rho(t, u) du \text{ as } N \rightarrow \infty \text{ in prob.}$$

where $\rho(t, u)$ is the unique solution of the Burger's equation:

$$\partial_t \rho(t, u) = -\partial_u \{ \rho(t, u)(1 - \rho(t, u)) \}$$

$$\begin{aligned} \partial_t \rho(t, u) &\sim \mathbb{E}[N L \eta_x^s] = N \mathbb{E}[\eta_{x-1}^s (1 - \eta_x^s) - \eta_x^s (1 - \eta_{x+1}^s)] \\ &= -\mathbb{E}[\partial^N \{ \eta_x^s (1 - \eta_{x+1}^s) \}] \rightarrow -\partial_u \{ \rho(t, u)(1 - \rho(t, u)) \} \end{aligned}$$

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Hamiltonian dynamics

Try to consider a Hamiltonian dynamics as a microscopic model

One-dimensional chain of oscillators:

- $(p_i, q_i)_{i \in \mathbb{T}_N} \in (\mathbb{R}^2)^N$: state space (p : momentum, q : displacement)
- $H(p, q) = \sum_i \frac{p_i^2}{2} + V(q_{i+1} - q_i) + U(q_i)$: Hamiltonian
- V : interaction potential, smooth, positive, $0 < C_1 \leq V'' \leq C_2 < \infty$
- U : pinning potential, smooth, positive

$$\begin{cases} \dot{q}_i = p_i \\ \dot{p}_i = V'(q_{i+1} - q_i) - V'(q_i - q_{i-1}) - U'(q_i) \end{cases}$$

One-dimensional chain of oscillators

We assume $U = 0$ and change the coordinates $r_i := q_i - q_{i-1}$ (deformation)

- $(p_i, r_i)_{i \in \mathbb{T}_N} \in (\mathbb{R}^2)^N$: state space
- $H(p, r) = \sum_i \frac{p_i^2}{2} + V(r_i)$: Hamiltonian

$$\begin{cases} \dot{r}_i = p_i - p_{i-1} \\ \dot{p}_i = V'(r_{i+1}) - V'(r_i) \end{cases}$$

Under the dynamics, the following quantities are conserved:

- total energy $\sum_i \mathcal{E}_i$ where $\mathcal{E}_i = \frac{p_i^2}{2} + V(r_i)$
- total momentum $\sum_i p_i$
- total displacement $\sum_i r_i$

They should be macroscopic parameters.

Scaling limit

We want to show, for some scaling parameter $\theta(N)$,

$$\frac{1}{N} \sum_i \mathcal{E}_i(\theta(N)t) \delta_{\frac{i}{N}}(du) \rightarrow \mathcal{E}(t, u) du$$

$$\frac{1}{N} \sum_i p_i(\theta(N)t) \delta_{\frac{i}{N}}(du) \rightarrow p(t, u) du$$

$$\frac{1}{N} \sum_i r_i(\theta(N)t) \delta_{\frac{i}{N}}(du) \rightarrow r(t, u) du$$

where $\mathcal{E}(t, u)$, $p(t, u)$ and $r(t, u)$ evolve according to some system of **diffusion equations**

Scaling limit

But...

- Which order θ_N is the proper scaling ?
- Ergodicity ?
- How to show the law of large numbers (or CLT) **without randomness** ??

Actually, if $V(r) = r^2$, then the model is integrable and $\theta_N \neq N^2$, namely the transport of energy is not diffusive.

Scaling limit

But...

- Which order θ_N is the proper scaling ?
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Actually, if $V(r) = r^2$, then the model is integrable and $\theta_N \neq N^2$, namely the transport of energy is not diffusive.

- We add a **stochastic noise** to the dynamics
- Under the new dynamics, **only the energy** is conserved
- We aim to obtain a macroscopic diffusion equation of energy by hydrodynamic limit or equilibrium fluctuation

Stochastic model

Work with Stefano Olla (CEREMADE)

We consider a Markov process on $(\mathbb{R}^2)^N$ with an infinitesimal generator

$$Lf(p, r) = Af(p, r) + \gamma Sf(p, r)$$

where

$\gamma > 0$: *strength of the noise*

$$A = \sum_{i \in \mathbb{T}_N} (X_i - Y_{i,i+1}) : \textit{Hamiltonian part}$$

$$S = \frac{1}{2} \sum_{i \in \mathbb{T}_N} \{(X_i)^2 + (Y_{i,i+1})^2\} : \textit{Stochastic part}$$

$$Y_{i,j} = p_i \partial_{r_j} - V'(r_j) \partial_{p_i}, \quad X_i = Y_{i,i}, \quad N+1 \equiv 1$$

Stochastic model

The process on $(\mathbb{R}^2)^N$ generated by $N^2 L$ can also be described by the following system of stochastic differential equations (SDEs)

$$\begin{cases} dp_i(t) = N^2[V'(r_{i+1}) - V'(r_i) - \frac{\gamma p_i}{2}\{V''(r_i) + V''(r_{i+1})\}]dt \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad + \sqrt{\gamma}N\{V'(r_{i+1})dB_i^1 - V'(r_i)dB_i^2\} \\ dr_i(t) = N^2[p_i - p_{i-1} - \gamma V'(r_i)]dt + \sqrt{\gamma}N\{-p_{i-1}dB_{i-1}^1 + p_i dB_i^2\} \end{cases}$$

where $\{B_i^1, B_i^2\}_{i \in \mathbb{T}_N}$ are $2N$ -independent standard Brownian motions.

In particular, if $\gamma = 0$, L defines the original Hamiltonian dynamics.

Conserved quantity and microcanonical surface

L conserves the total energy $\sum_i \mathcal{E}_i = H$ since

$$X_i H = 0, \quad Y_{i,i+1} H = 0$$

L does not conserve the total momentum $\sum_i p_i$ nor total length $\sum_i r_i$

The movement is constrained on the *microcanonical* surface of constant energy

$$\Sigma_{N,E} = \left\{ (p, r) \in (\mathbb{R}^2)^N; \frac{1}{N} \sum_{i=1}^N \mathcal{E}_i = \frac{1}{N} \sum_{i=1}^N \frac{p_i^2}{2} + V(r_i) = E \right\}.$$

- Our conditions on V assure that these surfaces are connected
- The vector fields $\{X_i, Y_{i,i+1}, i = 1, \dots, N\}$ are tangent to this surface
- $\text{Lie}\{X_i, Y_{i,i+1}, i = 1, \dots, N\}$ generates the all tangent space

Consequently the *microcanonical* measures $\nu_{N,E}(\cdot) = \nu_e^N(\cdot | \Sigma_{N,E})$ are ergodic for our dynamics.

Equilibrium fluctuation

Define the time dependent distribution

$$Y_t^N = \frac{1}{\sqrt{N}} \sum_i \delta_{i/N} \{ \mathcal{E}_i(N^2 t) - e \}$$

Theorem (Olla, S, 2011, PTRF)

If the process starts from the equilibrium measure ν_e^N , then Y_t^N converges in law to the solution of the linear SPDE

$$\partial_t Y = D(e) \Delta Y dt + \sqrt{2D(e)\chi(e)} \nabla B(u, t)$$

where B is the standard normalized space-time white noise.

$\chi(e)$ is the variance of \mathcal{E}_0 under the equilibrium measure ν_e and $D(e)$ is given by a complicated variational formula.

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Remark

If $V(r) = \frac{r^2}{2}$, then $D(e) = \frac{\gamma}{4} + \frac{1}{6\gamma}$.

Hydrodynamic limit

The last theorem almost implies the hydrodynamic limit holds for the energy distribution and the limiting equation is

$$\partial_t e(t, u) = \partial_u (D(e(t, u)) \partial_u e(t, u)),$$

but we need some more technical estimates (which seem hard to prove rigorously).

Key estimates for the proof

- Sharp estimates of **the spectral gap** for $-S = -S_N$ in the size of N as a linear operator on $L^2(\nu_{N,E})$
 - $E[(f - E[f])^2] \leq \lambda_{N,E}^{-1} E[f(-S_N)f], \quad \lambda_{N,E} \geq \text{const} \cdot N^{-2}$
- Sector condition
 - $E[fAg]^2 \leq CE[f(-S)f]E[g(-S)g]$
- Characterization of **“closed form” in the infinite dimensional space**
 - Closed form on $\Sigma_{N,E}$ is exact form! We use this fact, to show that, in the infinite dimensional space,
“closed forms” = “exact forms” + finite dimensional space

Related works (I)

There are many works for “*d*-dimensional chain of oscillators + noise”, but other models of the type “Hamiltonian system + noise” are rarely studied.

Topics:

- ergodicity (sufficient condition for the noise is known, without noise case is open problem)
- energy transport is diffusive or superdiffusive (it depends on the noise, but how it depends is not known)
- HDL under hyperbolic-scaling

Related works (II)

Results of hydrodynamic limit or equilibrium fluctuation under the diffusive scaling limit are very few:

Bernardin, Lyon, 2007 A model with **energy and length conserving noise** and $V(r) = \frac{r^2}{2}$, the limiting system of equations is

$$\begin{cases} \partial_t r(t, u) = \Delta r(t, u) \\ \partial_t e(t, u) = \Delta e(t, u) \end{cases}$$

Simon, Lyon, 2013 A model with **another energy and length conserving noise** and $V(r) = \frac{r^2}{2}$, the limiting system of equations is

$$\begin{cases} \partial_t r(t, u) = \Delta r(t, u) \\ \partial_t e(t, u) = \Delta \left(e(t, u) + \frac{r(t, u)^2}{2} \right) \end{cases}$$

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From deterministic microscopic dynamics

Is it impossible to start from purely deterministic microscopic dynamics??

→ No!

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→ No!

Two-step approach (Gaspard-Gilbert, 2008, 2009)

Microscopic deterministic Newtonian (Hamiltonian) dynamics

limit in some sense* →

Mesoscopic stochastic process of energy

HD limit →

Macroscopic diffusion equation (deterministic)

(* weak interaction limit, rare interaction limit)

Typical setting

N -particle system :

Microscopic level (**mechanical model**)

$\mathbf{q}_i \in \mathbb{R}^d$: i -th particle's position, $\mathbf{p}_i \in \mathbb{R}^d$: i -th particle's velocity

$(\mathbf{q}_i, \mathbf{p}_i)_{i=1}^N \in \mathbb{R}^{2dN}$: State space

Time evolution : deterministic, nearest-neighbor interaction

Equilibrium measure : $(\mathbf{p}_i)_{i=1}^N \sim$ **N -product of $\mathcal{N}(0, \beta^{-1} I_d)$**

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Time evolution : deterministic, nearest-neighbor interaction

Equilibrium measure : $(\mathbf{p}_i)_{i=1}^N \sim$ **N -product of $\mathcal{N}(0, \beta^{-1}I_d)$**

Mesoscopic level (**stochastic energy process**)

$e_i \in \mathbb{R}_+$: i -th particle's energy

$e_i = \frac{1}{2} \sum_{q=1}^d (p_i^{(q)})^2$ where $\mathbf{p}_i = (p_i^{(1)}, p_i^{(2)}, \dots, p_i^{(d)})$: kinetic energy

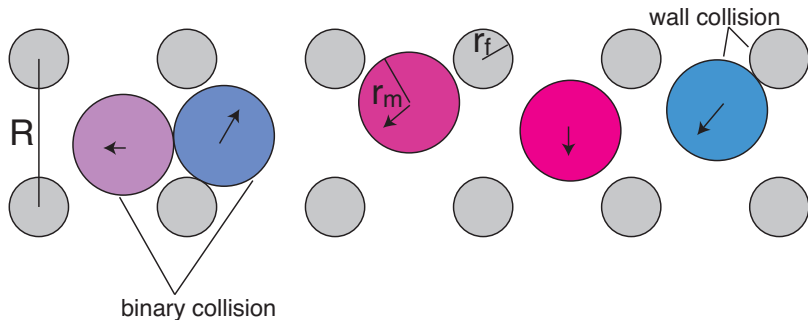
$(e_i)_{i=1}^N \in \mathbb{R}_+^N$: State space

Time evolution : stochastic, nearest-neighbor interaction

Equilibrium measure : $(e_i)_{i=1}^N \sim$ **N -product of $\Gamma(\frac{d}{2}, \beta^{-1})$**

Example 1 : Localized hard balls in 2 or 3 dimensions

Gaspard-Gilbert (2008,2009)

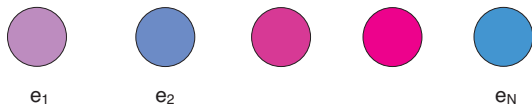


- Confined condition: $r_f + r_m > \frac{R}{2}$
- Binary Collision condition: $\epsilon := r_m - \sqrt{(r_f + r_m)^2 - (\frac{R}{2})^2} > 0$
- Energy transfer **only** occurs by binary collisions
- Take the **rare interaction limit** (i.e. $\epsilon \rightarrow 0$)

Rare interaction limit

In the limit $\epsilon \rightarrow 0$ where $r := r_f + r_m$ is fixed,

- equilibrium characterized by energy of the ball is achieved in each cell



- $(e_i)_{i=1}^N$ represents each state of the mesoscopic system
- Master equation for the probability $P_N(e_1, e_2, e_3, \dots, e_N; t)$ is derived (equivalently the infinitesimal generator is derived)

$$\mathcal{L}f(e) = \sum_{i=1}^{N-1} \Lambda_{GG}(e_i, e_{i+1}) \int P_{GG}(e_i, e_{i+1}, d\alpha) [f(T_{i,i+1,\alpha} e) - f(e)]$$

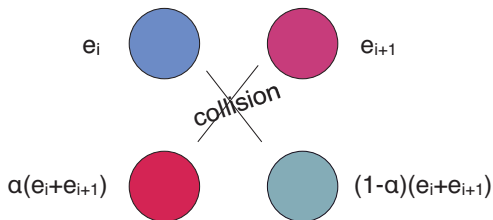
Mesososcopic dynamics

- Two neighboring balls having energy e_i and e_{i+1} collide with rate

$$\Lambda_{GG3}(e_i, e_{i+1}) = \frac{\sqrt{2\pi}}{6} \frac{(2e_i + e_{i+1}) \vee (e_i + 2e_{i+1})}{\sqrt{e_i \vee e_{i+1}}}$$

- When a collision occurs, new energy configuration becomes $(\alpha(e_i + e_{i+1}), (1 - \alpha)(e_i + e_{i+1}))$ with probability

$$P_{GG3}(e_i, e_{i+1}, d\alpha) = \frac{3}{2} \frac{1 \wedge \sqrt{\frac{\alpha \wedge (1-\alpha)}{\beta \wedge (1-\beta)}}}{\frac{1}{2} + \beta \vee (1-\beta)} d\alpha, \quad \beta = \frac{e_i}{e_i + e_{i+1}}$$



Example 2 : Energy transfer in a fast-slow Hamiltonian system

Dolgopyat-Liverani (2011)

Microscopic dynamics : N weakly coupled geodesic flows on d -dimensional manifolds of negative curvature with coupling strength $\epsilon > 0$ ($d \geq 3$)

weak interaction limit ($\epsilon \rightarrow 0$) \rightarrow

Mesoscopic dynamics : SDEs of energies

$$de_i = \sum_{j \in \mathbb{Z}^d; |i-j|=1} \beta(e_i(t), e_j(t)) dt + \sigma(e_i(t), e_j(t)) dB_{i,j}$$

where $\beta(a, b)$ and $\sigma(a, b)$ are given implicitly.

This is the only rigorous result for the first step !

Tasks

- Introduce general models describing the mesoscopic energy evolutions obtained by examples
- Prove hydrodynamic limit to derive **diffusion equation** of heat conduction from these models

Answer for the task 1

- **Stochastic energy exchange model** introduced by Grigo-Khanin-Szász (2011)
- **Energy conserving stochastic Ginzburg-Landau model** introduced by Stefano-Liverani (2012)

Stochastic energy exchange model

- $x = (x_i)_{i=1}^N \in \mathbb{R}_+^N$: state space
- x_i : energy of particle at site i
- $\Lambda(\cdot, \cdot) (> 0)$: rate of energy exchange (or collision), continuous
- $P(\cdot, \cdot, d\alpha)$: probability measure on $[0, 1]$ (collision kernel), continuous

$\{x(t)\}_{t \geq 0}$: Markov process on \mathbb{R}_+^N with generator \mathcal{L} acting on bounded functions $f : \mathbb{R}_+^N \rightarrow \mathbb{R}$ is

$$\mathcal{L}f(x) = \sum_{i=1}^{N-1} \Lambda(x_i, x_{i+1}) \int P(x_i, x_{i+1}, d\alpha) [f(T_{i,i+1,\alpha}x) - f(x)]$$

$$(T_{i,i+1,\alpha}x)_k = \begin{cases} \alpha(x_i + x_{i+1}) & \text{if } k = i \\ (1 - \alpha)(x_i + x_{i+1}) & \text{if } k = i + 1 \\ x_k & \text{if } k \neq i, i + 1 \end{cases}$$

Hydrodynamic limit

Formally, the expected statement of HD limit is

$$\frac{1}{N} \sum_{i=1}^N x_i(N^2 t) \delta_{\frac{i}{N}}(du) \rightarrow e(t, u) du \quad (N \rightarrow \infty), u \in [0, 1]$$

where $e(t, u)$ is the solution of $\partial_t e = \nabla(D(e)\nabla e)$ and the diffusion coefficient $D(e)$ is characterized by terms of Λ and P .

- For general (Λ, P) or (β, σ) , the system is of non-gradient type
- First step of the proof of HD limit for non-gradient system is to give a sharp estimate of the spectral gap of the generator \mathcal{L}

Spectral gap

- Total energy is conserved
- $\mathcal{S}_{e,N} := \{x \in \mathbb{R}_+^N ; \frac{1}{N} \sum_{i=1}^N x_i = e\}$: invariant
- Spectral of $-\mathcal{L}|_{\mathcal{S}_{e,N}}$ is our interest
- Any constant function is an eigenfunction associated with the eigenvalue 0
- For each microcanonical surface $\mathcal{S}_{e,N}$, there exists at least one invariant probability measure $\pi_{e,N}$

Spectral gap for reversible process

Assume that $\pi_{e,N}$ is **reversible measure** of $x(t)$ on $\mathcal{S}_{e,N}$

Dirichlet form associated with $\pi_{e,N}$:

$$\mathcal{D}_{e,N}(f) := \int \pi_{e,N}(dx) (-\mathcal{L}f)(x) f(x) = E_{\pi_{e,N}}[f(-\mathcal{L}f)]$$

Spectral gap of $-\mathcal{L}|_{\mathcal{S}_{e,N}}$ is characterized by

$$\lambda(e, N) := \inf_f \left\{ \frac{\mathcal{D}_{e,N}(f)}{E_{\pi_{e,N}}[f^2]} \mid E_{\pi_{e,N}}[f] = 0, f \in L^2(\pi_{e,N}) \right\}.$$

Assumptions

We assume that following typical properties for the models originated from Hamiltonian dynamics

- **Reversible with respect to the product gamma distribution** with some parameter $\gamma > 0$
- There exists a nice **scaling relation** : $\Lambda(ca, cb) = c^m \Lambda(a, b)$ and $P(ca, cb, \dots) = P(a, b, \dots)$ for all $c > 0$ with some $m \geq 0$

γ and m in examples

Gaspard-Gilbert model:

- $d = 3$ case : $\gamma = \frac{3}{2}$, $m = \frac{1}{2}$
- $d = 2$ case : $\gamma = \frac{2}{2} = 1$, $m = \frac{1}{2}$

Remark

$m \neq 0$ implies that $\Lambda(a, b)$ is not uniformly positive in $a, b > 0$. It makes the sharp estimate of the spectral gap quite hard.

Theorem (S,2013,submitted)

Under the assumption, $\exists C = C(m, \gamma) > 0$ s.t. $\forall N \geq 2$ and $\forall e > 0$,

$$\lambda(e, N) \geq C \lambda(1, 2) \frac{e^m}{N^2}$$

Corollary

For GG models in 2 or 3 dimensions, there exists a positive constant C such that,

$$\lambda(e, N) \geq C \frac{\sqrt{e}}{N^2}.$$

- If we can characterize **infinite and finite dimensional “closed form”**, then almost done
- Formally, under the assumption of the main theorem, the macroscopic equation should be

Stochastic energy exchange model

$$\partial_t e = \text{const.} \Delta(e^{m+1})$$

universal !!

Summary

- HDL or EF were mainly considered for interacting particle systems (interacting random walks)
- Derivation of energy diffusion via HDL or EF for **stochastic models originated from Hamiltonian dynamics** is hot topic !
- New rigorous techniques are generated quite recently !
- We need to understand **geometric properties of finite and infinite canonical surfaces**