Stochastic approach to a space-time scaling limit for Hamiltonian systems

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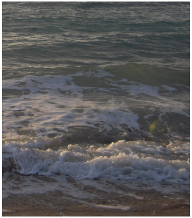
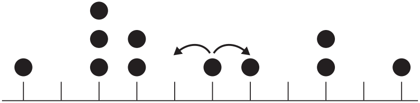
From microscopic world to macroscopic world





Motivation of our study:

"Explain macroscopic phenomena from microscopic dynamics"

In particular, a derivation of diffusion phenomena is the main interest of this talk

Micro and Macro

	Micro	Macro
Physical quantity	position or velocity of each molecular	density, temperature, pressure etc
Degree of freedom	enormous	a few
Time evolution	complicated interaction of elements	PDE

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- Each microscopic quantity has not (almost) any information for the macroscopic state
- Statistics (average, fluctuation etc) of microscopic quantities determines the macroscopic state
- These properties have good compatibility with stochastic analysis

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Deterministic or Stochastic?

If the microscopic dynamics is a deterministic classical dynamical system, then ...

- The microscopic dynamics is time reversible, but the diffusion equation is not!
- As $N \sim 10^{23}$ or more, it is impossible to solve or even simulate the dynamics
- It is very difficult to justify that we can forget about "microscopic quantities" in the limit

We often consider a stochastic process as a microscopic model and study scaling limits for the process

- Hydrodynamic limit
- Equilibrium fluctuation

Hydrodynamic limit and Equilibrium fluctutaion

Hydrodynamic limit is...

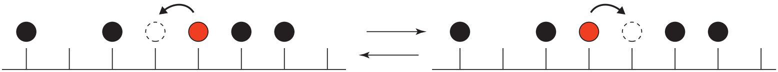
- rigorous method to derive deterministic macroscopic PDEs
- law of large numbers (LLN)

Equilibrium fluctuation is...

- rigorous method to derive stochastic macroscopic PDEs
- central limit theorem (CLT)

Example. Symmetric Simple Exclusion Process (SSEP)

- Continuous time symmetric random walks with hard core interaction



$\xrightarrow{\textit{HDL limit}}$ Properly scaled density of particles ρ and its fluctuation Y evolve according to

$$\partial_t \rho = \frac{1}{2}\Delta\rho, \quad dY_t = \frac{1}{2}\Delta Y_t dt + \sqrt{\rho_t(1-\rho_t)}\nabla dB_t$$

1. Typical stochastic models
    - Symmetric Simple Exclusion Process (SSEP)
    - Totally Asymmetric Simple Exclusion Process (TASEP)

2. Hamiltonian dynamics + stochastic noise

3. Two-step approach
    - Stochastic energy exchange model
    - Energy conserving stochastic Ginzburg-Landau model

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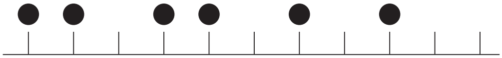
State space of SSEP and TASEP

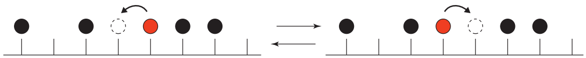
$\mathbb{Z}^d$: d-dimensional discrete lattice

$\chi^d := \{0,1\}^{\mathbb{Z}^d}$: state space, $\quad\quad \eta = (\eta_x)_{x \in \mathbb{Z}^d}$: element of χ^d

- $\eta_x = 1$: there is a particle at x
- $\eta_x = 0$: there is no particle at x



Symmetric Simple Exclusion Processes (SSEP)



- Continuous time Markov process on χ^d
- jump to one of the neighboring sites with probability $\frac{1}{2d}$
- jump rate (the inverse of the expectation value of random waiting time) is a constant 1
- exclusion rule

$\{\eta^t\}_{t\geq 0}$: Markov process on χ^d with generator

$$(Lf)(\eta) = \frac{1}{2d} \sum_{|x-y|=1} 1_{\{(\eta_x,\eta_y)=(1,0)\}}(f(\eta^{x\to y}) - f(\eta))$$

$Lf = \frac{d}{dt}P_t f|_{t=0}$, $\quad P_t$: Markov semigroup, $\quad P_t f(\eta) = E_\eta[f(\eta^t)]$

The number of particles is a unique conserved quantity
$\Rightarrow$ The density of particles characterizes the equilibrium (macroscopic) states ($\{\nu_\rho\}_{\rho\in[0,1]}$: Bernoulli product measures)
$\Rightarrow$ Derive an evolution equation of the density of particles

For simplicity, we consider the discrete torus $\mathbb{T}_N^d = \{1, 2, \dots, N\}^d$ instead of $\mathbb{Z}^d$ from now on.
Denote by π_t^N the scaled empirical measure under diffusive scaling:

$$\pi_t^N(du) = \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \eta_x(N^2 t)\delta_{\frac{x}{N}}(du) \in \mathcal{M}(\mathbb{T}^d = [0,1)^d)$$

($\mathcal{M}(\mathbb{T}^d = [0,1)^d)$: set of measures on $\mathbb{T}^d$)

Hydrodynamic limit for SSEP

Theorem (De Masi, et al. 1984)

Assume

$$\pi_0^N(du) \to \pi_0(du) = \rho_0(u)du \quad N \to \infty \quad \textit{in prob}$$

with some measurable function $\rho_0 : \mathbb{T}^d \to [0,1]$.
Then, $\forall t > 0$,

$$\pi_t^N(du) \to \pi_t(du) = \rho(t,u)du \quad N \to \infty \quad \textit{in prob}$$

where $\rho(t,u)$ *is the unique solution of the heat equation:*

$$\begin{cases} \partial_t \rho(t,u) = \dfrac{1}{2d} \Delta \rho(t,u) \\ \rho(0,\cdot) = \rho_0(\cdot) \end{cases}$$

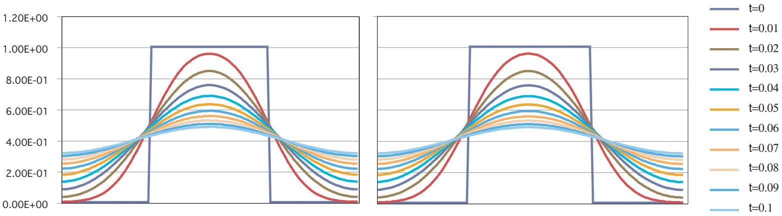
How to guess the limiting equation

Assume $\rho(t,u) \sim \mathbb{E}[\eta^s_x]$ for $u = \frac{x}{N}$ and $t = \frac{s}{N^2}$

$$\begin{aligned}\partial_t \rho(t,u) &\sim \mathbb{E}[N^2 L \eta^s_x] \\ &= \frac{N^2}{2d} \sum_{i=1}^{d} \mathbb{E}[\eta^s_{x+e_i}(1-\eta^s_x) + \eta^s_{x-e_i}(1-\eta^s_x) \\ &- \eta^s_x(1-\eta^s_{x+e_i}) - \eta^s_x(1-\eta^s_{x-e_i})] = \frac{N^2}{2d} \sum_{i=1}^{d} \mathbb{E}[\eta^s_{x+e_i} - 2\eta^s_x + \eta^s_{x-e_i}] \\ &\sim \frac{1}{2d} \Delta^N \rho(t,u) \xrightarrow{N\to\infty} \frac{1}{2d} \Delta \rho(t,u)\end{aligned}$$

$\Delta^N H(u) = \sum_{i=1}^{d} N^2 \{H(u + \frac{e_i}{N}) - 2H(u) + H(u - \frac{e_i}{N})\}$

Numerical simulation

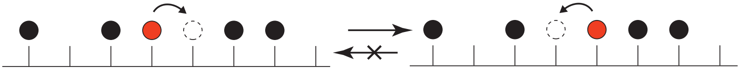


*exact solution of heat eq.

*numerical simulation of SSEP
$N = 100$
averaged density of $5,000,000$ paths

Totally Asymmetric Simple Exclusion Process (TASEP)

$\mathbb{Z}$: 1-dimensional discrete lattice
$\chi := \{0,1\}^{\mathbb{Z}}$: state space $\qquad \eta = (\eta_x)_{x\in\mathbb{Z}}$: element of χ



- jump only to right
- jump rate is a constant 1
- exclusion rule

$$(Lf)(\eta) = \sum_{x\in\mathbb{Z}} 1_{\{(\eta_x,\eta_{x+1})=(1,0)\}}(f(\eta^{x\to x+1}) - f(\eta))$$

Hydrodynamic limit for TASEP

Denote by π_t^N the scaled empirical measure under hyperbolic scaling:

$$\pi_t^N(du) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta_x(Nt)\delta_{\frac{x}{N}}(du) \in \mathcal{M}(\mathbb{R})$$

Theorem (Rezakhanlou, 1991)

Assume some conditions for π_0^N. Then, $\forall t > 0$,

$$\pi_t^N(du) \to \pi_t(du) = \rho(t,u)du \text{ as } N \to \infty \text{ in prob.}$$

where $\rho(t,u)$ is the unique solution of the Burger's equation:

$$\partial_t \rho(t,u) = -\partial_u\{\rho(t,u)(1-\rho(t,u))\}$$

$$\begin{aligned}\partial_t \rho(t,u) \sim \mathbb{E}[NL\eta_x^s] &= N\mathbb{E}[\eta_{x-1}^s(1-\eta_x^s) - \eta_x^s(1-\eta_{x+1}^s)] \\ &= -\mathbb{E}[\partial^N\{\eta_x^s(1-\eta_{x+1}^s)\}] \to -\partial_u\{\rho(t,u)(1-\rho(t,u)\}\end{aligned}$$

Hamiltonian dynamics

Try to consider a Hamiltonian dynamics as a microscopic model

One-dimensional chain of oscillators:

- $(p_i, q_i)_{i \in \mathbb{T}_N} \in (\mathbb{R}^2)^N$: state space (p : momentum, q: displacement)
- $H(p, q) = \sum_i \frac{p_i^2}{2} + V(q_{i+1} - q_i) + U(q_i)$: Hamiltonian

- V : interaction potential, smooth, positive, $0 < C_1 \leq V'' \leq C_2 < \infty$
- U : pinning potential, smooth, positive

$$\begin{cases} \dot{q}_i = p_i \\ \dot{p}_i = V'(q_{i+1} - q_i) - V'(q_i - q_{i-1}) - U'(q_i) \end{cases}$$

One-dimensional chain of oscillators

We assume $U = 0$ and change the coordinates $r_i := q_i - q_{i-1}$ (deformation)

- $(p_i, r_i)_{i \in \mathbb{T}_N} \in (\mathbb{R}^2)^N$: state space
- $H(p, r) = \sum_i \frac{p_i^2}{2} + V(r_i)$: Hamiltonian

$$\begin{cases} \dot{r}_i = p_i - p_{i-1} \\ \dot{p}_i = V'(r_{i+1}) - V'(r_i) \end{cases}$$

Under the dynamics, the following quantities are conserved:

- total energy $\sum_i \mathcal{E}_i$ where $\mathcal{E}_i = \frac{p_i^2}{2} + V(r_i)$
- total momentum $\sum_i p_i$
- total displacement $\sum_i r_i$

They should be macroscopic parameters.

Scaling limit

We want to show, for some scaling parameter $\theta(N)$,

$$\frac{1}{N}\sum_i \mathcal{E}_i(\theta(N)t)\delta_{\frac{i}{N}}(du) \to \mathcal{E}(t,u)du$$
$$\frac{1}{N}\sum_i p_i(\theta(N)t)\delta_{\frac{i}{N}}(du) \to p(t,u)du$$
$$\frac{1}{N}\sum_i r_i(\theta(N)t)\delta_{\frac{i}{N}}(du) \to r(t,u)du$$

where $\mathcal{E}(t,u), p(t,u)$ and $r(t,u)$ evolve according to some system of diffusion equations

Scaling limit

But...

- Which order θ_N is the proper scaling ?
- Ergodicity ?
- How to show the law of large numbers (or CLT) without randomness ??

Actually, if $V(r) = r^2$, then the model is integrable and $\theta_N \neq N^2$, namely the transport of energy is not diffusive.

Scaling limit

But...

- Which order θ_N is the proper scaling ?
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Actually, if $V(r) = r^2$, then the model is integrable and $\theta_N \neq N^2$, namely the transport of energy is not diffusive.

- We add a stochastic noise to the dynamics
- Under the new dynamics, only the energy is conserved
- We aim to obtain a macroscopic diffusion equation of energy by hydrodynamic limit or equilibrium fluctuation

Stochastic model

Work with Stefano Olla (CEREMADE)

We consider a Markov process on $(\mathbb{R}^2)^N$ with an infinitesimal generator

$$Lf(p,r) = Af(p,r) + \gamma Sf(p,r)$$

where

$$\begin{aligned} \gamma > 0 \ &: \ \textit{strength of the noise} \\ A = \sum_{i \in \mathbb{T}_N} (X_i - Y_{i,i+1}) \ &: \ \textit{Hamiltonian part} \\ S = \frac{1}{2} \sum_{i \in \mathbb{T}_N} \{(X_i)^2 + (Y_{i,i+1})^2\} \ &: \ \textit{Stochastic part} \end{aligned}$$

$$Y_{i,j} = p_i \partial_{r_j} - V'(r_j)\partial_{p_i}, \qquad X_i = Y_{i,i}, \quad N+1 \equiv 1$$

Stochastic model

The process on $(\mathbb{R}^2)^N$ generated by $N^2 L$ can also be described by the following system of stochastic differential equations (SDEs)

$$\begin{cases} dp_i(t) = N^2[V'(r_{i+1}) - V'(r_i) - \dfrac{\gamma p_i}{2}\{V''(r_i) + V''(r_{i+1})\}]dt \\ \qquad\qquad\qquad\qquad\qquad\qquad + \sqrt{\gamma} N\{V'(r_{i+1})dB_i^1 - V'(r_i)dB_i^2\} \\ dr_i(t) = N^2[p_i - p_{i-1} - \gamma V'(r_i)]dt + \sqrt{\gamma} N\{-p_{i-1}dB_{i-1}^1 + p_i dB_i^2\} \end{cases}$$

where $\{B_i^1, B_i^2\}_{i \in \mathbb{T}_N}$ are $2N$-independent standard Brownian motions.

In particular, if $\gamma = 0$, L defines the original Hamiltonian dynamics.

Conserved quantity and microcanonical surface

L conserves the total energy $\sum_i \mathcal{E}_i = H$ since

$$X_i H = 0, \quad Y_{i,i+1} H = 0$$

L does not conserve the total momentum $\sum_i p_i$ nor total length $\sum_i r_i$

The movement is constrained on the *microcanonical* surface of constant energy

$$\Sigma_{N,E} = \left\{ (p,r) \in (\mathbb{R}^2)^N ; \frac{1}{N} \sum_{i=1}^N \mathcal{E}_i = \frac{1}{N} \sum_{i=1}^N \frac{p_i^2}{2} + V(r_i) = E \right\}.$$

- Our conditions on V assure that these surfaces are connected
- The vector fields $\{X_i, Y_{i,i+1}, i = 1, \dots, N\}$ are tangent to this surface
- $\mathrm{Lie}\{X_i, Y_{i,i+1}, i = 1, \dots, N\}$ generates the all tangent space

Consequently the *microcanonical* measures $\nu_{N,E}(\cdot) = \nu_e^N(\cdot | \Sigma_{N,E})$ are ergodic for our dynamics.

Equilibrium fluctuation

Define the time dependent distribution

$$Y_t^N = \frac{1}{\sqrt{N}} \sum_i \delta_{i/N} \left\{ \mathcal{E}_i(N^2 t) - e \right\}$$

Theorem (Olla,S, 2011, PTRF)

If the process starts from the equilibrium measure ν_e^N, then Y_t^N converges in law to the solution of the linear SPDE

$$\partial_t Y = D(e) \triangle Y \, dt + \sqrt{2D(e)\chi(e)} \, \nabla B(u, t)$$

where B is the standard normalized space-time white noise.

$\chi(e)$ is the variance of $\mathcal{E}_0$ under the equilibrium measure ν_e and $D(e)$ is given by a complicated variational formula.

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Remark

If $V(r) = \frac{r^2}{2}$, then $D(e) = \frac{\gamma}{4} + \frac{1}{6\gamma}$.

Hydrodynamic limit

The last theorem almost implies the hydrodynamic limit holds for the energy distribution and the limiting equation is

$$\partial_t e(t, u) = \partial_u(D(e(t, u))\partial_u e(t, u)),$$

but we need some more technical estimates (which seem hard to prove rigorously).

Key estimates for the proof

- Sharp estimates of the spectral gap for $-S = -S_N$ in the size of N as a linear operator on $L^2(\nu_{N,E})$
  - $E[(f - E[f])^2] \leq \lambda_{N,E}^{-1} E[f(-S_N)f], \quad \lambda_{N,E} \geq const.N^{-2}$
- Sector condition
  - $E[fAg]^2 \leq CE[f(-S)f]E[g(-S)g]$
- Characterization of "closed form" in the infinite dimensional space
  - Closed form on $\Sigma_{N,E}$ is exact form! We use this fact, to show that, in the infinite dimensional space,

    "closed forms" $=$ "exact forms" $+$ finite dimensional space

Related works (I)

There are many works for "d-dimensional chain of oscillators + noise", but other models of the type "Hamiltonian system + noise" are rarely studied.

Topics:

- ergodicity (sufficient condition for the noise is known, without noise case is open problem)
- energy transport is diffusive or superdiffusive (it depends on the noise, but how it depends is not known)
- HDL under hyperbolic-scaling

Related works (II)

Results of hydrodynamic limit or equilibrium fluctuation under the diffusive scaling limit are very few:

Bernardin, Lyon, 2007 A model with energy and length conserving noise and $V(r) = \frac{r^2}{2}$, the limiting system of equations is

$$\begin{cases} \partial_t r(t,u) = \triangle r(t,u) \\ \partial_t e(t,u) = \triangle e(t,u) \end{cases}$$

Simon, Lyon, 2013 A model with another energy and length conserving noise and $V(r) = \frac{r^2}{2}$, the limiting system of equations is

$$\begin{cases} \partial_t r(t,u) = \triangle r(t,u) \\ \partial_t e(t,u) = \triangle \Big(e(t,u) + \dfrac{r(t,u)^2}{2} \Big) \end{cases}$$

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From deterministic microscopic dynamics

Is it impossible to start from purely deterministic microscopic dynamics??
$\rightarrow$ No!

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$\rightarrow$ No!

Two-step approach (Gaspard-Gilbert, 2008, 2009)

Microscopic deterministic Newtonian (Hamiltonian) dynamics

$\xrightarrow{\textit{limit in some sense}*}$

Mesoscopic stochastic process of energy

$\xrightarrow{\textit{HD limit}}$

Macroscopic diffusion equation (deterministic)

(* weak interaction limit, rare interaction limit)

Typical setting

N-particle system :

Microscopic level (mechanical model)

$\mathbf{q}_i \in \mathbb{R}^d$: i-th particle's position, $\mathbf{p}_i \in \mathbb{R}^d$: i-th particle's velocity

$(\mathbf{q}_i, \mathbf{p}_i)_{i=1}^N \in \mathbb{R}^{2dN}$: State space

Time evolution : deterministic, nearest-neighbor interaction

Equilibrium measure : $(\mathbf{p}_i)_{i=1}^N \sim$ N-product of $\mathcal{N}(0, \beta^{-1} I_d)$

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$(\mathbf{q}_i, \mathbf{p}_i)_{i=1}^N \in \mathbb{R}^{2dN}$: State space
Time evolution : deterministic, nearest-neighbor interaction
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Mesoscopic level (stochastic energy process)

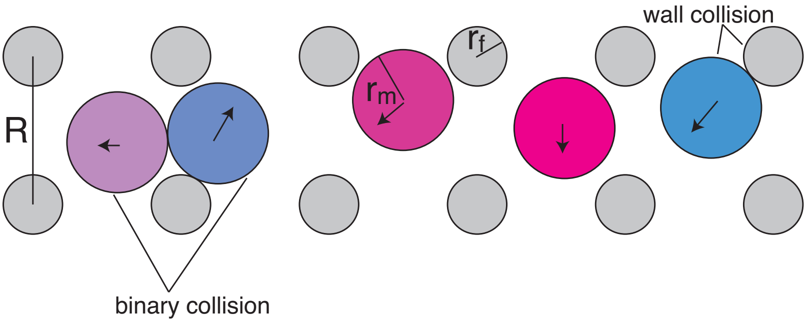
$e_i \in \mathbb{R}_+$: i-th particle's energy

$e_i = \frac{1}{2}\sum_{q=1}^{d}(p_i^{(q)})^2$ where $\mathbf{p}_i = (p_i^{(1)}, p_i^{(2)}, \dots, p_i^{(d)})$: kinetic energy

$(e_i)_{i=1}^N \in \mathbb{R}_+^N$: State space
Time evolution : stochastic, nearest-neighbor interaction
Equilibrium measure : $(e_i)_{i=1}^N \sim$ N-product of $\Gamma(\frac{d}{2}, \beta^{-1})$

Example 1 : Localized hard balls in 2 or 3 dimensions

Gaspard-Gilbert (2008,2009)

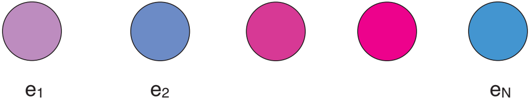


- Confined condition: $r_f + r_m > \frac{R}{2}$
- Binary Collision condition: $\epsilon := r_m - \sqrt{(r_f + r_m)^2 - (\frac{R}{2})^2} > 0$
- Energy transfer only occurs by binary collisions
- Take the rare interaction limit (i.e. $\epsilon \to 0$)

Rare interaction limit

In the limit $\epsilon \to 0$ where $r := r_f + r_m$ is fixed,

- equilibrium characterized by energy of the ball is achieved in each cell



- $(e_i)_{i=1}^N$ represents each state of the mesoscopic system
- Master equation for the probability $P_N(e_1, e_2, e_3, \ldots, e_N; t)$ is derived (equivalently the infinitesimal generator is derived)

$$\mathcal{L}f(e) = \sum_{i=1}^{N-1} \Lambda_{GG}(e_i, e_{i+1}) \int P_{GG}(e_i, e_{i+1}, d\alpha)[f(T_{i,i+1,\alpha}e) - f(e)]$$

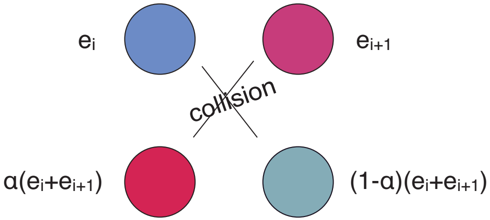
Mesoscopic dynamics

- Two neighboring balls having energy e_i and e_{i+1} collide with rate

$$\Lambda_{GG3}(e_i, e_{i+1}) = \frac{\sqrt{2\pi}}{6} \frac{(2e_i + e_{i+1}) \vee (e_i + 2e_{i+1})}{\sqrt{e_i \vee e_{i+1}}}$$

- When a collision occurs, new energy configuration becomes $(\alpha(e_i + e_{i+1}), (1 - \alpha)(e_i + e_{i+1}))$ with probability

$$P_{GG3}(e_i, e_{i+1}, d\alpha) = \frac{3}{2} \frac{1 \wedge \sqrt{\frac{\alpha \wedge (1-\alpha)}{\beta \wedge (1-\beta)}}}{\frac{1}{2} + \beta \vee (1 - \beta)} d\alpha, \quad \beta = \frac{e_i}{e_i + e_{i+1}}$$



Example 2 : Energy transfer in a fast-slow Hamiltonian system

Dolgopyat-Liverani (2011)

Microscopic dynamics : N weakly coupled geodesic flows on d-dimensional manifolds of negative curvature with coupling strength $\epsilon > 0$ ($d \geq 3$)

$\xrightarrow{\textit{weak interaction limit } (\epsilon \to 0)}$

Mesoscopic dynamics : SDEs of energies

$$de_i = \sum_{j \in \mathbb{Z}^d; |i-j|=1} \beta(e_i(t), e_j(t))dt + \sigma(e_i(t), e_j(t))dB_{i,j}$$

where $\beta(a, b)$ and $\sigma(a, b)$ are given implicitly.
This is the only rigorous result for the first step !

Our interest

Tasks

- Introduce general models describing the mesoscopic energy evolutions obtained by examples
- Prove hydrodynamic limit to derive diffusion equation of heat conduction from these models

Answer for the task 1

- Stochastic energy exchange model introduced by Grigo-Khanin-Szász (2011)
- Energy conserving stochastic Ginzburg-Landau model introduced by Stefano-Liverani (2012)

Stochastic energy exchange model

- $x = (x_i)_{i=1}^N \in \mathbb{R}_+^N$: state space
- x_i : energy of particle at site i
- $\Lambda(\cdot,\cdot)(>0)$: rate of energy exchange (or collision), continuous
- $P(\cdot,\cdot,d\alpha)$: probability measure on $[0,1]$ (collision kernel), continuous

$\{x(t)\}_{t\geq 0}$: Markov process on $\mathbb{R}_+^N$ with generator $\mathcal{L}$ acting on bounded functions $f : \mathbb{R}_+^N \to \mathbb{R}$ is

$$\mathcal{L}f(x) = \sum_{i=1}^{N-1} \Lambda(x_i, x_{i+1}) \int P(x_i, x_{i+1}, d\alpha)[f(T_{i,i+1,\alpha}x) - f(x)]$$

$$(T_{i,i+1,\alpha}x)_k = \begin{cases} \alpha(x_i + x_{i+1}) & \text{if } k = i \\ (1-\alpha)(x_i + x_{i+1}) & \text{if } k = i+1 \\ x_k & \text{if } k \neq i, i+1 \end{cases}$$

Hydrodynamic limit

Formally, the expected statement of HD limit is

$$\frac{1}{N}\sum_{i=1}^{N} x_i(N^2 t)\delta_{\frac{i}{N}}(du) \to e(t,u)du \quad (N \to \infty), u \in [0,1]$$

where $e(t,u)$ is the solution of $\partial_t e = \nabla(D(e)\nabla e)$ and the diffusion coefficient $D(e)$ is characterized by terms of Λ and P.

- For general (Λ, P) or (β, σ), the system is of non-gradient type
- First step of the proof of HD limit for non-gradient system is to give a sharp estimate of the spectral gap of the generator $\mathcal{L}$

Spectral gap

- Total energy is conserved
- $\mathcal{S}_{e,N} := \{x \in \mathbb{R}_+^N \; ; \; \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} x_i = e\}$: invariant
- Spectral of $-\mathcal{L}|_{\mathcal{S}_{e,N}}$ is our interest
- Any constant function is an eigenfunction associated with the eigenvalue 0
- For each microcanonical surface $\mathcal{S}_{e,N}$, there exists at least one invariant probability measure $\pi_{e,N}$

Spectral gap for reversible process

Assume that $\pi_{e,N}$ is reversible measure of $x(t)$ on $\mathcal{S}_{e,N}$

Dirichlet form associated with $\pi_{e,N}$:

$$\mathcal{D}_{e,N}(f) := \int \pi_{e,N}(dx)(-\mathcal{L}f)(x)f(x) = E_{\pi_{e,N}}[f(-\mathcal{L}f)]$$

Spectral gap of $-\mathcal{L}|_{\mathcal{S}_{e,N}}$ is characterized by

$$\lambda(e,N) := \inf_f \left\{ \frac{\mathcal{D}_{e,N}(f)}{E_{\pi_{e,N}[f^2]}} \middle| E_{\pi_{e,N}}[f] = 0,\ f \in L^2(\pi_{e,N}) \right\}.$$

Assumptions

We assume that following typical properties for the models originated from Hamiltonian dynamics

- Reversible with respect to the product gamma distribution with some parameter $\gamma > 0$
- There exists a nice scaling relation : $\Lambda(ca, cb) = c^m \Lambda(a, b)$ and $P(ca, cb, \dots) = P(a, b, \dots)$ for all $c > 0$ with some $m \geq 0$

γ and m in examples

Gaspard-Gilbert model:

- $d = 3$ case : $\gamma = \frac{3}{2}$, $m = \frac{1}{2}$
- $d = 2$ case : $\gamma = \frac{2}{2} = 1$, $m = \frac{1}{2}$

Remark

$m \neq 0$ implies that $\Lambda(a, b)$ is not uniformly positive in $a, b > 0$. It makes the sharp estimate of the spectral gap quite hard.

Spectral gap estimate

Theorem (S,2013,submitted)

Under the assumption, $\exists C = C(m,\gamma) > 0$ *s.t.* $\forall N \geq 2$ *and* $\forall e > 0$,

$$\lambda(e,N) \geq C\lambda(1,2)\frac{e^m}{N^2}$$

HD limit

Corollary

For GG models in 2 or 3 dimensions, there exists a positive constant C such that,

$$\lambda(e, N) \geq C\frac{\sqrt{e}}{N^2}.$$

- If we can characterize infinite and finite dimensional "closed form", then almost done
- Formally, under the assumption of the main theorem, the macroscopic equation should be

Stochastic energy exchange model

$$\partial_t e = const.\Delta(e^{m+1})$$

universal !!

Summary

- HDL or EF were mainly considered for interacting particle systems (interacting random walks)
- Derivation of energy diffusion via HDL or EF for stochastic models originated from Hamiltonian dynamics is hot topic !
- New rigorous techniques are generated quite recently !
- We need to understand geometric properties of finite and infinite canonical surfaces