

Projective transformation groups of pseudo-Riemannian manifolds

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Geometry and Dynamics
Tokyo, September 16, 2013

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Geometry and Dynamics

(a biased point of view?)

Meaning of “Geometry and Dynamics”?

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Goal: Put them in competition and see what happens \rightarrow Classify

Gromov's vague conjecture

“**Rigid** *Geometric Structures* on compact manifolds with *Large* automorphism group are **Classifiable!**”

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Any example of them plays a central role in its category, e.g. as is the sphere in conformal geometry...

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“**Rigid** *Geometric Structures* on compact manifolds with *Large* automorphism group are **Classifiable!**”

Any example of them plays a central role in its category, e.g. as is the sphere in conformal geometry...

There is a (mathematical) definition of “geometric structure” and “rigid” but none for “large” and “classifiable”

Definition of Rigidity, by examples

Rigid: (solid)

Riemannian metric

Pseudo-Riemannian metric

Affine connection

Definition of Rigidity, by examples

Rigid: (solid)

Riemannian metric

Pseudo-Riemannian metric

Affine connection

Non-rigid: (fluid)

Symplectic structure

Contact structure

Complex structure

A plane field

Steps (in working with the conjecture)

0. Pick a geometric structure (or a class of them).
1. State a precise conjecture: important and non-trivial step
2. Prove it.

Projective structures

- Connection ∇ : it allows derivation of vector fields on M
- Geometrically: a second order differential equation on $M \rightarrow$ a vector field \mathcal{G}_∇ on M whose orbits are the geodesics of ∇

There is equivalence $\nabla \iff \mathcal{G}_\nabla$

∇ is determined by \mathcal{G}_∇

- Projective structure (or projective connection?): a class of connections having the same non-parametrized geodesics

Morphisms

A diffeomorphism $(M, \nabla) \rightarrow (M', \nabla')$ is projective if it sends unparameterized geodesics of ∇ to unparameterized geodesics of ∇'

Metric projective structures

g Riemannian metric

∇_g its Levi-Cevita connection

σ_g the projective structure associated to ∇_g

Levi-Civita connection exists in the pseudo-Riemannian case (although there is no direct variational definition of geodesics)..

Two metrics g and \bar{g} on M are projectively equivalent if Id_m is projective between ∇_g and $\nabla_{\bar{g}}$

Automorphisms groups

- $\text{Aut}(M, g)$: isometries of g : $\text{Iso}(M, g)$
- $\text{Aut}(M, \nabla_g)$: diffeomorphisms preserving (parameterized) geodesics of g , classically called $\text{Aff}(M, g)$
- $\text{Aut}(M, \sigma_g)$: diffeomorphisms preserving unparameterized geodesics of g , classically $\text{Proj}(M, g)$

Clearly: $\text{Iso}(M, g) \subset \text{Aff}(M, g) \subset \text{Proj}(M, g)$

Experimental result

Our first main result is to give a precise statement of Gromov's conjecture for metric projective structures:

Conjecture (Projective Lichnerowicz conjecture)

Let (M, g) be a compact pseudo-Riemannian manifold.

Then $\text{Proj}(M, g)/\text{Aff}(M, g)$ is finite unless (M, g) is isometrically covered by the standard Riemannian sphere (up to constant)?

(The new fact here is to consider the full groups instead of their identity components...)

Mathematical Results

Theorem (The conjecture is true in the Riemannian and in the analytic-Lorentz cases)

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Theorem (The conjecture is true in the Riemannian and in the analytic-Lorentz cases)

Let (M, g) be a compact pseudo-Riemannian manifold.

- *If g is Riemannian, then $\text{Proj}(M, g)/\text{Aff}(M, g)$ is finite, unless M is a (finite) quotient of the standard sphere.*
- *If g is Lorentzian and analytic, then $\text{Proj}(M, g)/\text{Aff}(M, g)$ is finite.*

All objects in a simple case

Geodesics

\mathbb{R}^n Euclidean space

(Parameterized) geodesic: a curve with constant speed and whose support is a straight line segment

$$\gamma :]a, b[\rightarrow \mathbb{R}^n, \frac{d^2\gamma}{dt^2} = 0$$

Non-parameterized geodesic: γ has its image in a straight segment \iff there exists function $p(t)$ such $\frac{d^2\gamma}{dt^2} = p(t) \frac{d\gamma}{dt}$

Transformations

U, V open sets in \mathbb{R}^n , $f : U \rightarrow V$ diffeomorphism,

f **isometric** if it preserves distances,

Then, $\exists A \in O(n, (\mathbb{R}))$, $b \in \mathbb{R}^n$, such that $f =$ restriction to U of the map $x \rightarrow Ax + b$

f **affine** if it sends (a parameterized) geodesic to a (parametrized) geodesic.

Then, $\exists A \in GL_n(\mathbb{R})$, $b \in \mathbb{R}^n$, such that $f =$ restriction to U of the map $x \rightarrow Ax + b$

f **projective** if it sends (a unparameterized) geodesic to a (unparameterized) geodesic.
(f preserves geometric line segments)

Fundamental Theorem of projective real geometry

A projective transformation (between open sets of the Euclidean space) is a homography:

There exist $\alpha_0, \dots, \alpha_n$ linear forms β_0, \dots, β_n scalars

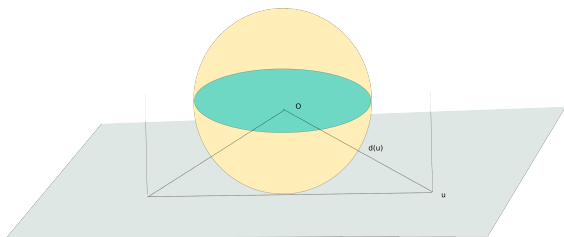
$$f(x) = \left(\frac{\alpha_1(x) + \beta_1}{\alpha_0(x) + \beta_0}, \dots, \frac{\alpha_n(x) + \beta_n}{\alpha_0(x) + \beta_0} \right)$$

Seen on the sphere

The standard sphere \mathbb{S}^n is **projectively flat**: any point has a neighbourhood projectively diffeomorphic to an open set in \mathbb{R}^n

Precisely: any (open) hemisphere is (globally) projectively diffeomorphic to \mathbb{R}^n : a perspective map...

perspective



PGL_{n+1}

- Any local projective transformation of \mathbb{S}^n extends globally to \mathbb{S}^n (\mathbb{S}^n is an equivariant projective compactification of \mathbb{R}^n)

$$\text{Proj}(\mathbb{S}^n) = \text{PGL}_{n+1}(\mathbb{R})$$

$$(\text{= } \text{GL}_{n+1}(\mathbb{R})/\mathbb{R}^* = \text{SL}_{n+1}(\mathbb{R}) \text{ up to index 2})$$

$$A \in \text{GL}_{n+1}(\mathbb{R}) \text{ acts as } A.x = \frac{Ax}{\|Ax\|}$$

It preserves linear 2-planes of \mathbb{R}^{n+1} and hence great circles of \mathbb{S}^n

Alternatively, in the projective space

PGL_{n+1} acts naturally on $\mathbb{P}^n(\mathbb{R}) = \mathbb{R}^{n+1} - \{0\}/\mathbb{R}^*$

If $\mathbb{P}^n(\mathbb{R}) = \mathbb{S}^n / \pm Id$ is endowed with the quotient metric, then affine charts (in \mathbb{R}^n) are projective (and not affine)

Beltrami Theorem

The hyperbolic space too is projectively flat:

because of the Klein model (a complete metric on the unit ball of curvature -1 whose geodesics are line segments)

Beltrami: *A locally projectively flat Riemannian manifold has constant sectional curvature*

Some remarks

Regularity: we assumed transformations f diffeomorphic, but the theorem is true for f merely bijective (without measurability hypothesis...)

Complex case: f sends an affine complex line to an affine complex line (precisely open sets in them)

Same conclusion: complex homography

Regularity

Regularity is not automatic in the complex case?!

There is a bijection $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ sending any complex algebraic curve defined by a polynomial of degree d , to a similar one (in particular preserving affine complex lines), but f non continuous...

Some historical facts

What is new?

- In the Riemannian case, the theorem is proved by V. Matveev, but for the identity components:

$\text{Proj}^0(M, g) = \text{Aff}^0(M, g)$, unless M is covered by the sphere,

In other words

- **Hypothesis:** (M, g) admits a projective non-affine one parameter group of transformations

- **Conclusion** M is a quotient of the sphere

Here:

- **Hypothesis:** (M, g) has a projective transformation none of which powers is affine

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Conclusion M is a quotient of the sphere

●● The Lorentz case (in the analytic case) is new: only partial (technical) results are known.

The projective group: a classical Problem

A Russian speciality

Current names: Bolsinov (UK), Topalov (?), Matveev (Germany)

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Italian:

Beltrami, Dini, Fubini, Levi-Civita....

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Older: Solodovnikov, Sinjukov, Aminova,

Italian:

Beltrami, Dini, Fubini, Levi-Civita....

Others,

Weyl, Eisenhart, Painlevé, Darboux, Lagrange, Cartan, Lie,...

Hall (relativity)

Bryant,

Kähler version

A Japanese speciality: Hasegawa, Fujimura, Ishihara, Yano, Hiramatu, Yoshimatsu...

Here, one defines h-planar curves and h-projective transformations accordingly,

A curve $\gamma : [a, b] \rightarrow M$ is h-planar if its complexified tangent direction field is parallel:

$T(t) = \mathbb{C}\gamma'(t)$ is parallel along γ

Theorem

Let (M, g) be a compact Kähler manifold.

Then $\text{Proj}^{\text{hol}}(M, g)$ is a finite extension of $\text{Aff}^{\text{hol}}(M, g)$, unless (M, g) is (isometrically and holomorphically) covered by $\mathbb{P}^n(\mathbb{C})$ endowed with its Fubini-Study metric (up to a constant).

The identity component case “ $\text{Proj}^{\text{hol}^0} = \text{Aff}^{\text{hol}^0}$ unless (M, g) is covered by the projective space”, was proved by Matveev and Rosemann..

Further remarks, Projective vs Projective

- In complex geometry: M is projective if it embeds holomorphically in $\mathbb{P}^N(\mathbb{C})$,
say $M \subset \mathbb{P}^N(\mathbb{C})$

- $G_M = \{A \in \text{PGL}_{N+1}(\mathbb{C}) / A(M) = M\}$ could be called the projective group of M !?

It may be large, e.g non-compact and acts transitively

- On the other hand, Induce the Fubini-Study metric on M : $(M, g_{FS|_M})$

$\text{Proj}^{\text{hol}}(M, g_{FS|_M}) = ?$

Theorem

$\text{Proj}^{\text{hol}}(M, g_{FS|_M})$ is a finite extension of $\text{Iso}^{\text{hol}}(M, g_{FS|_M})$ unless M is a Veronese submanifold.

Corollary: for non Veronese submanifolds G_M does not act projectively!

Proof: by our theorem if $\text{Proj}^{\text{hol}}(M, g_{FS|_M})$ is not a finite extension of $\text{Iso}^{\text{hol}}(M, g_{FS|_M})$, then $(M, g_{FS|_M})$ is isometric to $(\mathbb{P}^n(\mathbb{C}), cg_{FS})$, for some constant c , $n = \dim M$ (up to a cover, but forget it)

Nash \cap Kodaira

Question: Find holomorphic and isometric immersions
 $(\mathbb{P}^n(\mathbb{C}), c\mathcal{G}_{FS}) \rightarrow (\mathbb{P}^N(\mathbb{C}), \mathcal{G}_{FS})$

Answer: up to composition with an ambient isometry (in SU_{N+1}), this is a Veronese map:

$$v_k : [X_0, \dots, X_n] \rightarrow [\dots X^k \dots]$$

where X^k ranges over all monomials of degree k in X_0, \dots, X_n .

- In particular $c = k$ (quantic fact!)

Calabi Egregium Theorem

Theorem

(Extrinsic = Intrinsic)

Let $F(N, b)$ denote the simply connected Hermitian space of dimension N and constant holomorphic sectional curvature b .

Let M be a Kähler manifold (not necessarily complete) and $f : M \rightarrow F(N, b)$ a holomorphic isometric immersion.

Then, f is rigid in the sense that any other immersion f' is deduced from f by composing with an element of $\text{Iso}(F(N, b))$

Example: the induced metric on an elliptic curve can never be flat (but almost)!

Same fact but by other methods for Calabi-Yau manifolds... (but Fano could be at some scalings?)

Idea

(Calabi's thesis)

Diastasis (diastatic function) $D(p, q)$

Geometry creates Dynamics

(back to motivations...)

- M a smooth manifold:

- Action of $\text{Diff}(M)$ on $\text{Met}(M)$ its space of metrics...

(M, g) Riemannian (or pseudo-) \rightarrow

- Geodesic flow

- Symmetry groups, e.g. $\text{Iso}(M, g)$, $\text{Aff}(M, g)$, $\text{Proj}(M, g)$,
But they are generically trivial.

Goal: Look for and characterize the special non-generic cases!

(like $\text{SL}(n, \mathbb{Z})$ in groups!)

Hierarchy of Groups

(M, g) pseudo-Riemannian

- $\text{Iso}(M, g)$ the group of all isometries, i.e. diffeomorphisms such that $f^*g = g$
- $\text{Aff}(M, g)$... preserving parametrized geodesics of (M, g) .
- $\text{Proj}(M, g)$... preserving unparameterized geodesics of (M, g) .

Hierarchy of Groups

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– $\text{Proj}(M, g)$... preserving unparameterized geodesics of (M, g) .

– $\text{Sim}(M, g)$ the group of similarities (or homotheties) i.e. maps f such that $f^*g = ag$ for some constant.

– $\text{Conf}(M, g)$ the group of conformal transformations such that $f^*g = ag$, where a is a function on M .

Inclusions

$\text{Iso} \subset \text{Sim} \subset \text{Conf}$,

$\text{Iso} \subset \text{Sim} \subset \text{Aff} \subset \text{Proj}$.

Special spaces are those for which inclusion is non-trivial

Focus today on the question: when is $\text{Aff} \subsetneq \text{Proj}$?

Remark: they are all Lie groups with identity components $\text{Iso}^0, \dots, \text{Aff}^0, \text{Proj}^0$.

Example 1: Conformal beauty of the sphere (a central substratum)

- Observation: The sphere is beautiful, meaning: the inclusion chain is non-trivial!

$$\text{Iso}(\mathbb{S}^n) = O(n + 1), \text{Conf}(\mathbb{S}^n) = O(1, n + 1)$$

Example 1: Conformal beauty of the sphere (a central substratum)

- Observation: The sphere is beautiful, meaning: the inclusion chain is non-trivial!

$$\text{Iso}(\mathbb{S}^n) = O(n + 1), \text{Conf}(\mathbb{S}^n) = O(1, n + 1)$$

- Rigidity: the sphere is uniquely beautiful, Lichnerowicz conjecture (solved by Ferrand and Obata): \mathbb{S}^n is the unique compact Riemannian manifold for which the conformal group is not equal to the isometry group of any metric in its conformal class?

Example 2: Projective beauty of the sphere

$$\text{Proj}(\mathbb{S}^n) = \text{PGL}_{n+1}(\mathbb{R})$$

$$(\text{= } \text{GL}_{n+1}(\mathbb{R})/\mathbb{R}^* = \text{SL}_{n+1}(\mathbb{R}) \text{ up to index 2})$$

$$A \in \text{GL}_{n+1}(\mathbb{R}), A \cdot x = \frac{Ax}{\|Ax\|}$$

Alternatively,

$$\text{PGL}_{n+1} \text{ acts on } \mathbb{P}^n(\mathbb{R}) = \mathbb{R}^{n+1} - \{0\}/\mathbb{R}^*$$

$$\text{Aff}(\mathbb{S}^n) = \text{Iso}(\mathbb{S}^n)$$

Some finite quotients of the sphere may have Proj non-compact,

- Our question here: projective rigidity of the sphere?

Example 3: Affine beauty of the torus

$\mathbb{T}^n = \mathbb{R}^n / \Lambda$, Λ lattice in \mathbb{R}^n , e.g. $\Lambda = \mathbb{Z}^n$

Iso = \mathbb{T}^n , up to a finite index

Aff = $GL_n(\mathbb{Z})$ (= $SL_n(\mathbb{Z})$ up to index 2),

But $\text{Proj}(\mathbb{T}^n) = \text{Aff}(\mathbb{T}^n)$

- Affine rigidity of the torus is not easy to state...

Projective flatness

$f : (M, g) \rightarrow (M', g')$ projective diffeomorphism

A metric (M, g) is (locally) projectively flat if it is projectively diffeomorphic to the Euclidean space.

Betrami: in this case (M, g) has constant sectional curvature

One also defines projectively flat (non-metric) connections

This leads to a (G, X) -structure with $G = \mathrm{PGL}_{n+1}(\mathbb{R})$, $X = \mathbb{P}^n(\mathbb{R})$

Terminology: “projective structure” is sometimes used to mean a “flat projective structure” ...

Space of metrics, equivalence relations, without group actions

Restrict discussion to the Riemannian case \rightarrow generalize to pseudo-...

$\mathcal{M}et(M)$ be the space of all Riemannian metrics on M .

(projective) Equivalence relation: $g \sim g'$ iff they have the same unparameterized geodesics

$\mathcal{M}et^{\text{Proj}}(g)$ = equivalence class of g

$\mathcal{M}et^{\text{Proj}}(g) \supset \mathbb{R}g$

Let $g \rightarrow \nabla_g$ (its Levi-Civita connection) $\rightarrow \sigma_g$ (associated projective structure), then the classes are levels of $g \rightarrow \sigma_g$

When $\text{Diff}(M)$ acts on $\text{Met}(M)$, the stabilizer of g is $\text{Iso}(M, g)$, and the stabilizer of the (projective) class of g is $\text{Proj}(M, g)$.

The action of $\text{Proj}(M, g)$ on $\text{Met}^{\text{Proj}}(g)$, is a priori, neither trivial, nor transitive... (it may happen that $\text{Proj}(M, g)$ is trivial but not is $\text{Met}^{\text{Proj}}(g)$).

Remark: affine, conformal... equivalence classes can be defined similarly.

Rank

Fundamental Fact: $\text{Met}^{\text{Proj}}(g)$ has a finite dimension, bounded by that of the standard metric on the sphere of same dimension.

$\dim \text{Met}^{\text{Proj}}(g) = \text{Degree of (projective) mobility of } (M, g)$
 ≥ 1 (since $\text{Met}^{\text{Proj}}(g) \supset \mathbb{R}g$)

Rank = degree of mobility -1

Hint in the affine case:

Write $\bar{g}_x(u, v) = g_x(T_x(u), v)$,

$x \rightarrow T_x \in \text{End}(T_x M)$ an endomorphism of TM (symmetric with respect to g)

Write $\bar{g} = gT$ and $T = g^{-1}\bar{g}$

If \bar{g} affinely equivalent to g ,

then \bar{g} is parallel with respect to ∇_g , hence T is parallel (w.r.t. ∇_g)

Hence $\dim \text{Met}^{\text{Aff}}(g) \leq n(n+1)/2$, $n = \dim M$

(Details for the projective case latter on)

Philosophy

Let $f \in \text{Diff}(M)$ act naturally on $\text{Met}(M)$

- The f - action has a fixed point $\iff f$ is an isometry for some Riemannian metric on M .

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QUESTION What is the dynamical counterpart of the fact that the f -action preserves some (finite dimensional) manifold V in $\text{Met}(M)$. (special case $\dim V = 2$)

Reminiscent of mapping class group action on the Teichmuller space...

Metric on the space of metrics

$g \in \text{Met}(M)$,

$T_g \text{Met}(M) = \text{space of vector fields } \chi(M)$

Endow $\chi(M)$ with the L_g^2 -metric

Special case $\dim \mathcal{M}et^{\text{Proj}}(M, g) = 2$

We need to consider only this case because in the higher rank case we have:

Theorem (Kiosak - Matveev, Mounoud)

Let (M, g) a compact pseudo-Riemannian that is not covered by the standard (Riemannian) sphere.

If the degree of projective mobility of (M, g) is ≥ 3 , then any projectively equivalent metric to g is affinely equivalent to it.

Challenge!

Corollary: all what remains to consider is the challenging case when $\dim \mathcal{M}et^{\text{Proj}}(M, g) = 2$

$\text{Proj}(M, g)$ acts on the surface $\mathcal{M}et^{\text{Proj}}(M, g)$

Challenge!

Corollary: all what remains to consider is the challenging case when $\dim \mathcal{M}et^{\text{Proj}}(M, g) = 2$

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Paradoxical thoughts:

→ The rank 1 case is easier to handle since Proj acts on a smaller space $\mathcal{M}et^{\text{Proj}}$?

← The higher rank case is easier since when $\mathcal{M}et^{\text{Proj}}$ is big then Proj will be big.

- In rank one case, one proves Proj is small, say generated by one diffeomorphism f
- Such f is dissipative, say a north-south dynamics with an attractor and repulser being submanifolds of some codimension.
- Associated are many geodesic and umbilical (but singular) foliations...
- But, how to detect that you are on a (rigorously) round sphere!?

Some similar apparently easy geometric problems:

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- Blaschke conjecture: every compact Riemannian manifold whose injectivity radius equals its diameter is a compact rank one symmetric space?

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- Blaschke conjecture: every compact Riemannian manifold whose injectivity radius equals its diameter is a compact rank one symmetric space?
- Warning: co-existence of “dissipative” and chaotic dynamics: there exist Riemannian metrics on compact 3-manifolds whose geodesic flow is completely integrable but chaotic !

(Ref: Bolsinov, Taimanov: Integrable geodesic flows with positive topological entropy)

(Both work on the projective equivalence and projective transformation problems)

Some projective differential geometry

(emphasising on its non-linear character as is the classical Schwarzian derivative..., and avoiding Cartan connections theory...)

Examples

Dini

On surfaces, near a generic point, g and \bar{g} are projectively equivalent \iff in some co-ordinate system:

$$g = (X(x) - Y(y))(dx^2 + dy^2)$$

and

$$\bar{g} = \left(\frac{1}{Y(y)} - \frac{1}{X(x)} \right) \left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)} \right)$$

$$X(x) > Y(y)$$

Remarks

X and Y are (essentially) eigen-values of the tensor T defined by:
 $\bar{g}(\cdot, \cdot) = g(T\cdot, \cdot)$.

Observe the Commutation: $X(x, y) = X(x)$, $Y(x, y) = Y(y)$

In a domain with a Dini normal form: strong simplicity:

$X(x_1, y_1) > Y(x_2, y_2)$: $\inf X > \sup Y$

Compactness forces confluence of eigenvalues and leads to rigidity...

A projective transformation of the torus

On \mathbb{T}^2

$$g = \left(f(x) - \frac{1}{f(y)}\right)(\sqrt{f(x)}dx)^2 + \frac{1}{\sqrt{f(y)}}dy^2$$

$$\phi(x, y) = (y, x)$$

is projective, but not affine (except f very special)

Ellipsoid: a recent example, by means of integrable systems theory

(Topalov, Matveev, Tabachnikov)

Ellipsoid:

$$\sum_{i=1}^n \frac{(x_i)^2}{a_i} = 1$$

g = the metric induced from \mathbb{R}^n

$$\bar{g} = \frac{1}{\sum \left(\frac{x_i}{a_i}\right)^2} \left(\sum \frac{dx_i^2}{a_i} \right)$$

Affine equivalence

$g \rightarrow \nabla_g$ is non linear...

g and \bar{g} affinely equivalent

\iff

$\nabla_g = \nabla_{\bar{g}}$ (Levi-Civita connections)

Equation: $\nabla_g - \nabla_{\bar{g}} = 0$ is non-linear on g

$\bar{g}(u, v) = g(Tu, v)$, write $T = g^{-1}\bar{g}$

$T : TM \rightarrow TM$ endomorphism, a $(1, 1)$ -tensor

Linearization: $\nabla_g - \nabla_{\bar{g}} = 0 \iff T \text{ parallel} \iff \nabla_g T = 0$

Projective equivalence

$\nabla \rightarrow (\Gamma_{ij}^k)$ and $\bar{\nabla} \rightarrow (\bar{\Gamma}_{ij}^k)$ are projectively equivalent if they have the same unparametrized geodesics

The Geodesic equations $\ddot{x}^k = \Gamma_{ij}^k(x)\dot{x}^i\dot{x}^j$ and $\ddot{x}^k = \bar{\Gamma}_{ij}^k(x)\dot{x}^i\dot{x}^j$ have the same geometric solutions

$A = \nabla - \bar{\nabla}$ tensor, $A : TM \times TM \rightarrow TM$

Projective equivalence: A traceless, i.e. $A(u, u)$ is parallel to $u = 0$

$\iff A$ has the form: $A(u, v) = I(u)v + I(v)u$ for some form I

Non-linear equation on g

$$\nabla_u \bar{g}(\xi, \eta) = \bar{g}(\xi, \eta) d\theta(u) + \frac{1}{2} \bar{g}(\xi, u) d\theta(\eta) + \frac{1}{2} \bar{g}(\eta, u) d\theta(\xi)$$

$$\theta = \ln\left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{1+n}}$$

“Linearization!”

$$T = g^{-1}\bar{g}$$

Define L such that $T = \frac{L^{-1}}{\det L}$
 i.e. $\bar{g}(u, v) = \frac{1}{\det L} g(L^{-1}u, v)$ (write $\bar{g} = \frac{1}{\det L} gL^{-1}$), so,

$$L = \left(\frac{\det \bar{g}}{\det g} \right)^{\frac{1}{n+1}} \bar{g}^{-1} g$$

$T \rightarrow L$ is a (partially defined) transform on the sections of $End(TM)$

Proposition

g and \bar{g} are projectively equivalent $\iff L$ satisfies the linear equation:

$$g((\nabla_u L)v, w) = \frac{1}{2}g(v, u)d\text{trace}(L)(w) + \frac{1}{2}g(w, u)d\text{trace}(L)(v)$$

Say that such L is a \mathcal{P} -tensor, and denote their space $\mathcal{P}(M, g)$.

So, $\bar{g} \in \text{Met}^{\text{Proj}}(M, g) \rightarrow L \in \mathcal{P}(M, g)$ is a bijection onto its image (= the invertible L).

Remarks

1. For g and \bar{g} projectively equivalent
 $\bar{g} \in \text{Met}^{\text{Proj}}(M, g) \notin \text{Met}^{\text{Proj}}(M, \bar{g})$, but $\mathcal{P}(M, g) \neq \mathcal{P}(M, \bar{g})$
2. The $\text{Diff}(M)$ -action on the sections of $\text{End}(TM)$ obtained by composing with the $T \rightarrow L$ transform is still linear!
3. We have in particular a linear representation
 $\rho : \text{Proj}(M, g) \rightarrow \text{GL}(\mathcal{P}(M, g))$

Nijenhuis tensor

Elements of \mathcal{P} are not parallel, nevertheless have some special properties:
For L section of $\text{End}(TM)$,

$$N_L(u, v) = [Lu, Lv] - L[Lu, v] - L[u, Lv] - L^2[u, v]$$

A \mathcal{P} -tensor has $N_L = 0$

Integrability

An almost complex structure has a vanishing Nijenhuis $N_L = 0 \iff$ it is integrable, i.e. it defines a complex structure.

In general, if L diagonalizable,

- Eigen- distributions are integrable

(define $\lambda : M \rightarrow \mathbb{C}$ as eigenfunction it is continuous and $\lambda(x)$ is a spectral value of $L(x)$, $\forall x$)

- An eigenfunction is constant along the leaves of the other distributions
- In particular eigenfunctions with higher multiplicity (> 1) are constant
- L is symmetric (auto-adjoint w.r.t. g), thus eigen-distributions are orthogonal.
- L is parallel \iff all eigenfunctions are constant.

Foliations

If constant eigenvalue exist, then we get a festival of geodesic and umbilical... foliations, but singular.

In the pseudo-Riemannian non-Riemannian case, foliations may degenerate: the restriction of the metric on leaves is a degenerate.

Example: The weak stable foliation of the geodesic flow on negatively curved metric on a surface, is a lightlike geodesic foliation for some natural Lorentz metric on the unit tangent bundle.

Elements of proof, Actions

Stress tensor

(M, g) pseudo-Riemannian

$f \in \text{Diff}(M)$

Usual stress (or strength):

$$T_f = g^{-1} f^* g, \text{ (i.e. } f^* g(\cdot, \cdot) = g(T_f \cdot, \cdot))$$

Use the transform $T \rightarrow L$, define an adapted stress K_f as:

$$f^* g = \frac{1}{\det K_f} g K_f^{-1}$$

Action

$\text{Proj}(M, g)$ acts on $\text{Met}^{\text{Proj}}(M, g)$

$$(f, g) \in \text{Proj}(M, g) \times \text{Met}^{\text{Proj}}(M, g) \rightarrow f^*g$$

Transported action on $\mathcal{P}(M, g)$ via the map

$$L \rightarrow g_L = \frac{1}{\det L} g L^{-1}$$

$$(f, L) \in \text{Proj}(M, g) \times \mathcal{P}(M, g) \rightarrow f^*L.K_f \in \mathcal{P}(M, g)$$

The action is linear!

Representation

$$\rho : \text{Proj}(M, g) \rightarrow \text{GL}(\mathcal{P}(M, g)) = \text{GL}_2(\mathbb{R})$$

$$\rho(f) = 1 \iff K_f = I \iff f \text{ isometry}$$

$$\rho(f) \text{ homothety} \iff K_f = aI, f \text{ is a similarity, impossible if } M \text{ compact} \\ (\text{unless } a = \pm 1)$$

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Say

$$\rho : \text{Proj}(M, g) \rightarrow \text{SL}_2(\mathbb{R})$$

Homography

Fix f , $K = K_f$

$\{I, K\}$ a basis of $\mathcal{P}(M, g)$

$$\rho(f)K = f^*L.K = aI + bK$$

$$f^*K = \frac{aI + bK}{K}$$

Let

$$A = A_f = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$$

It acts homographically on \mathbb{C} :

$$A \cdot z = \frac{az + b}{z}$$

$\text{Sections}(\text{End}(TM))$ is an algebra, and $A \cdot$ acts on it

Previous equation $f^*K = A \cdot K$

Consequently

$$f^{n*}K = A^n \cdot K$$

(one knows the action of f on its stress tensor...)

Spectrum of K

$x \rightarrow Sp(x)$ spectrum of $K(x)$

$Sp(x)$ subset $\mathbb{C} \times \dots \mathbb{C} \dots$

$$Sp(f^n x) = A^n \cdot Sp(x)$$

If $\lambda : M \rightarrow \mathbb{R}$ is eigenfunction: $\lambda(x) \in Sp(x)$,

Up to taking a power,

$$\lambda(f^n) = A^n \cdot \lambda(x)$$

λ semi-conjugates the two dynamical systems $(M, f) \rightarrow (\mathbb{C}, A)$

Classification of elements of SL_2

elliptic

parabolic

hyperbolic

Let λ real eigenfunction

$\lambda(M) \subset \mathbb{R}$ is a compact A -invariant interval

Cases $\rho(f)$ elliptic or parabolic

- Elliptic and preserving an interval implies $A^2 = 1$
- Parabolic with an invariant interval implies $\lambda(M)$ is a fixed point.
- Hyperbolic case: if λ non-constant, then $\lambda(M) = [\lambda_-, \lambda_+]$
 λ_- and λ_+ fixed points of A .

South-North dynamics between them,

Hyperbolic case

(M, g) is Riemannian,

- K is diagonalizable

- As suggested by Dini normal form (in dimension 2), Matveev and Topolov prove that that a somewhere inequality between eigenfunctions extends to the whole M :

$$\lambda_1(x_0) < \lambda_2(x_0) \implies \sup \lambda_1 \leq \inf \lambda_2$$

- It follows that there exists exactly one non-constant eigenfunction λ with range $[\lambda_-, \lambda_+]$
- λ_- and λ_+ could be (constant) eigenfunctions,
- λ is simple since by the general theory (of Nijenhuis tensors) a non-simple eigenfunction is constant.

Weyl tensor

- All this allows understanding topological (i.e. without measure) Lyapunov exponents...
- \rightarrow vanishing of the projective Weyl tensor
- (M, g) is projectively flat.... \square

Lorentz case

- K is not, a priori, diagonalizable: an auto-adjoint endomorphism w.r.t. a non-definite non-degenerate form is not necessarily diagonalizable....
 - Global ordering of eigenfunctions is no longer valid.
 - The stress tensor has no dynamical meaning, e.g. $K = 1$ means f is isometric, but may have non-trivial Lyapunov exponents...!
- ... \square

Conclusion: Cultural Content

1. There is a Gromov's vague conjecture on classifiability of geometric substrata with a large symmetry group
2. Fundamental Theorem of projective geometry
3. Beltrami Theorem: projectively flat Riemannian metrics have constant sectional curvature
4. There are non measurable projective bijections of \mathbb{C}^n

5. Experimental Fact (= conjecture): besides the standard Riemannian sphere, and up to finite objects, any projective transformation of a compact pseudo-Riemannian manifold is affine.
6. There is Theorema Egregium of Calabi for holomorphic-isometric immersions: extrinsic quantities turn out to be intrinsic.
7. There are geodesic flows of compact Riemannian manifolds which are completely integrable and chaotic!

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6. There is Theorema Egregium of Calabi for holomorphic-isometric immersions: extrinsic quantities turn out to be intrinsic.
7. There are geodesic flows of compact Riemannian manifolds which are completely integrable and chaotic!
8. Tokyo is a very beautiful geometric structure!