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Generators for the tautological algebra of the moduli space of curves

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GENERATORS FOR THE TAUTOLOGICAL ALGEBRA OF THE MODULI SPACE OF CURVES

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Abstract. In this paper, we prove that the tautological algebra *in cohomology* of the moduli space $\mathbf{M}_g$ of smooth projective curves of genus g is generated by the first $[g/3]$ Mumford-Morita-Miller classes. This solves a part of Faber's conjecture [5] concerning the structure of the tautological algebra affirmatively. More precisely, for any k we express the k-th Mumford-Morita-Miller class e_k as an explicit polynomial in the lower classes for all genera $g = 3k-1, 3k-2, \cdots, 2$.

1. Introduction

Let $\mathbf{M}_g$ be the moduli space of smooth projective curves of genus g and let $\kappa_i \in \mathcal{A}^*(\mathbf{M}_g)$ be the i-th tautological class introduced by Mumford [24] where $\mathcal{A}^*(\mathbf{M}_g)$ denotes the rational Chow algebra of $\mathbf{M}_g$. Let $\mathcal{R}^*(\mathbf{M}_g)$ be the subalgebra of $\mathcal{A}^*(\mathbf{M}_g)$ generated by the classes κ_i. It is called the tautological algebra of $\mathbf{M}_g$ (see Faber [5] and Looijenga [15]).

let $\mathcal{M}_g$ be the mapping class group of a closed oriented surface Σ_g of genus g. As is well known, there exists a close connection between the moduli space $\mathbf{M}_g$ and the mapping class group $\mathcal{M}_g$. In particular, we have a canonical isomorphism $H^*(\mathbf{M}_g;\mathbb{Q}) \cong H^*(\mathcal{M}_g;\mathbb{Q})$ for any $g \geq 2$. In [18][19] (see also [17]) we defined certain integral cohomology classes $e_i \in H^{2i}(\mathcal{M}_g;\mathbb{Z})$ $(i = 1, 2, \cdots)$ which serve as characteristic classes of surface bundles. It turned out that, over the rationals e_i corresponds to the projection of the class $(-1)^{i+1}\kappa_i$ to the rational cohomology under the above isomorphism. These classes are sometimes called the Mumford-Morita-Miller classes. It may be natural to define the tautological algebra $\mathcal{R}^*(\mathcal{M}_g)$ of the mapping class group to be the subalgebra of $H^*(\mathcal{M}_g;\mathbb{Q})$ generated by the classes e_i. Then there is a natural surjection $\mathcal{R}^*(\mathbf{M}_g) \to \mathcal{R}^*(\mathcal{M}_g)$ and we may call the latter the tautological algebra *in cohomology* of the moduli space.

In [5], Faber proposed a beautiful conjecture about the structure of the tautological algebra $\mathcal{R}^*(\mathbf{M}_g)$ based on numerous explicit computations (see also [6][9]). There have been obtained several pieces of evidence for Faber's conjecture. Looijenga [15] proved that $\mathcal{R}^*(\mathbf{M}_g)$ is trivial in degrees $> g-2$ and also that $\mathcal{R}^{g-2}(\mathbf{M}_g)$ is isomorphic to $\mathbb{Q}$ or 0. Then Faber [4] proved that $\mathcal{R}^{g-2}(\mathbf{M}_g)$ is in fact isomorphic to $\mathbb{Q}$.

Now the purpose of the present paper is to add one more evidence for Faber's conjecture. More precisely, it concerns the number of generators for the tautological algebra. Mumford [24] proved that $\mathcal{R}^*(\mathbf{M}_g)$ is generated by the classes

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$\kappa_1, \cdots, \kappa_{g-2}$. Then Faber [2][3] proved that for $g \geq 4$ the class κ_{g-2} can be expressed as a polynomial in the lower classes so that $\mathcal{R}^*(\mathbf{M}_g)$ is generated by the classes $\kappa_1, \cdots, \kappa_{g-3}$ for $g \geq 4$. He has some further results along these lines. Now it is a part of Faber's conjecture that $\mathcal{R}^*(\mathbf{M}_g)$ is already generated by the first $[\frac{g}{3}]$ classes $\kappa_1, \cdots, \kappa_{[\frac{g}{3}]}$. We prove this part of Faber's conjecture affirmatively *in cohomology*. We mention that the full conjecture implies that the projection $\mathcal{R}^*(\mathbf{M}_g) \to \mathcal{R}^*(\mathcal{M}_g)$ is an isomorphism. The following is the main theorem of the present paper.

Theorem 1.1. *The tautological algebra $\mathcal{R}^*(\mathcal{M}_g)$ in cohomology of the moduli space $\mathbf{M}_g$ of curves of genus g is generated by the first $[\frac{g}{3}]$ Mumford-Morita-Miller classes*

$$e_1, e_2, \cdots, e_{[\frac{g}{3}]} \in H^*(\mathcal{M}_g; \mathbb{Q}).$$

More precisely, for any positive integer k there are explicit polynomials

$$f_{k,j} \in \mathbb{Q}[e_1, \cdots, e_{k-1}] \quad (2 \leq j \leq 3k-1)$$

such that

$$e_k = f_{k,g}(e_1, \cdots, e_{k-1}) \in \mathcal{R}^{2k}(\mathcal{M}_g) \subset H^{2k}(\mathcal{M}_g; \mathbb{Q})$$

for all genera $g = 3k-1, 3k-2, \cdots, 2$.

The explicit polynomials $f_{k,g}$ mentioned in the above theorem will be given in Theorem 5.7, Theorem 6.5, Theorem 7.4 and Theorem 7.9 for $g = 3k-1, 3k-2, 3k-3, 3k-4$ respectively and in Theorem 8.5 for general $g \leq 3k-1$. We describe the first few cases separately because the general formula involves infinitely many sequences of natural numbers each of which is defined recursively in terms of previously defined ones so that it would be difficult to understand the geometrical background of the formula. Also in the first few cases, we compute our polynomials explicitly for low genera and we find that they are compatible with Faber's previous results mentioned in [5] whenever we can compare them.

The result of the present paper was announced in [23].

Remark 1.2. As is remarked in [5], Harer's result in [8], which improves his stability theorem in [7], implies that in lower (real) degrees $\leq 2[\frac{g}{3}]$ there are no relations in $\mathcal{R}^*(\mathcal{M}_g)$.

2. Cocycles for the Mumford-Morita-Miller classes

In this section, we summarize the results of our previous results from [22][11][12] (see also [23]) for later use.

Let $\mathcal{M}_{g,*}$ be the mapping class group of Σ_g relative to a base point. We have the universal Euler class $e \in H^2(\mathcal{M}_{g,*}; \mathbb{Z})$. Over the rationals, it corresponds to the negative of the first Chern class of the relative dualizing sheaf of the universal family of curves $\mathbf{C}_g \to \mathbf{M}_g$ over the moduli space. We define the tautological algebra $\mathcal{R}^*(\mathcal{M}_{g,*})$ of $\mathcal{M}_{g,*}$ to be the subalgebra of $H^*(\mathcal{M}_{g,*}; \mathbb{Q}) \cong H^*(\mathbf{C}_g; \mathbb{Q})$ generated by the classes $e_1, e_2, \cdots$ and e.

We denote simply by H the first homology group $H_1(\Sigma_g; \mathbb{Z})$ of Σ_g and set $H_\mathbb{Q} = H \otimes \mathbb{Q}$. Fix a symplectic basis of H so that the Siegel modular group $Sp(2g, \mathbb{Z})$ acts on H by automorphisms preserving the intersection pairing on it. In [21] we constructed representations

$$\begin{array}{ccc} \mathcal{M}_{g,*} & \xrightarrow{\rho_1} & \frac{1}{2}\Lambda^3 H \rtimes Sp(2g,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathcal{M}_g & \xrightarrow[\rho_1]{} & \frac{1}{2}\Lambda^3 H/H \rtimes Sp(2g,\mathbb{Z}). \end{array} \tag{1}$$

of the mapping class groups $\mathcal{M}_{g,*}, \mathcal{M}_g$ into certain groups which are semi-direct products of $Sp(2g,\mathbb{Z})$ with abelian groups $\frac{1}{2}\Lambda^3 H$ and $\frac{1}{2}\Lambda^3 H/H$. More explicitly ρ_1 in the top row is defined as

$$\rho_1(\varphi) = (\tilde{k}(\varphi), \rho_0(\varphi)) \quad (\varphi \in \mathcal{M}_{g,*})$$

where $\tilde{k} : \mathcal{M}_{g,*} \to \frac{1}{2}\Lambda^3 H$ is a certain crossed homomorphism whose cohomology class is uniquely defined and $\rho_0 : \mathcal{M}_{g,*} \to Sp(2g,\mathbb{Z})$ is the classical representation. The other ρ_1 in the bottom row is defined similarly. Here $\Lambda^3 H$ denotes the third exterior power of H and H is regarded as a natural submodule of $\Lambda^3 H$ by the embedding

$$H \ni u \longmapsto u \wedge \omega_0 \in \Lambda^3 H$$

where $\omega_0 \in \Lambda^2 H$ denotes the symplectic class. We set $U = \Lambda^3 H/H$ and $U_{\mathbb{Q}} = U \otimes \mathbb{Q}$. Since the targets of the above representations are semi-direct products, they induce the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}\big(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}), \mathbb{Q}\big)^{Sp} & \xrightarrow{\rho_1^*} & H^*(\mathcal{M}_{g,*};\mathbb{Q}) \\ \uparrow & & \uparrow \\ \mathrm{Hom}\big(\Lambda^* U_{\mathbb{Q}}, \mathbb{Q}\big)^{Sp} & \xrightarrow[\rho_1^*]{} & H^*(\mathcal{M}_g;\mathbb{Q}). \end{array} \tag{2}$$

where the symbol Sp in the left hand side indicates $Sp(2g,\mathbb{Q})$-invariant subspaces of various Sp-modules. Let $\mathcal{G}_{2k}$ denote the set of all isomorphism classes of *connected* trivalent graphs with $2k$ vertices and let $\mathcal{G}^0_{2k} \subset \mathcal{G}_k$ be the subset consisting of such graphs without loops. Here a loop means an edge whose two endpoints are the same vertex. We set $\mathcal{G}$ (resp. $\mathcal{G}^0$) to be the disjoint union of $\mathcal{G}_{2k}$ (resp. $\mathcal{G}^0_{2k}$) for all k. Then by making use of Weyl's classical representation, we constructed surjective homomorphisms

$$\begin{aligned} \mathbb{Q}[\Gamma;\Gamma \in \mathcal{G}] &\longrightarrow \mathrm{Hom}\big(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}), \mathbb{Q}\big)^{Sp} \\ \mathbb{Q}[\Gamma;\Gamma \in \mathcal{G}^0] &\longrightarrow \mathrm{Hom}\big(\Lambda^* U_{\mathbb{Q}}, \mathbb{Q}\big)^{Sp} \end{aligned} \tag{3}$$

from polynomial algebras generated by the graphs in $\mathcal{G}$ and $\mathcal{G}^0$ to the Sp-invariant parts of the dual spaces of the exterior algebras of $\Lambda^3 H_{\mathbb{Q}}$ and $U_{\mathbb{Q}}$. It turned out that these are specific cases of a general framework given by Kontsevich in [13][14]. By combining (2) and (3), we obtain the following commutative diagram

$$\begin{array}{ccc} \mathbb{Q}[\Gamma;\Gamma \in \mathcal{G}] & \xrightarrow{\Phi_{g,*}} & H^*(\mathcal{M}_{g,*};\mathbb{Q}) \\ \uparrow & & \uparrow \\ \mathbb{Q}[\Gamma;\Gamma \in \mathcal{G}^0] & \xrightarrow[\Phi_g]{} & H^*(\mathcal{M}_g;\mathbb{Q}) \end{array} \tag{4}$$

which is in fact defined at the cocycle level. We simply write $\alpha_\Gamma \in H^{2k}(\mathcal{M}_{g,*};\mathbb{Q})$ for $\Phi_{g,*}(\Gamma)$ and similarly write $\beta_\Gamma \in H^{2k}(\mathcal{M}_g;\mathbb{Q})$ for $\Phi_g(\Gamma)$ for any trivalent graph

Γ with $2k$ vertices which may not be connected. For completeness, we describe explicit formulae for the cocycles of $\mathcal{M}_{g,*}$ and $\mathcal{M}_g$ which represent α_Γ and β_Γ as definitions here (see [12] for details).

Definition 2.1. Let Γ be a trivalent graph with $2k$ vertices. Choose a linear chord diagram C with $6k$ vertices such that the associated trivalent graph Γ_C is isomorphic to Γ (see Definition 3.1 in §3). Then we set

$$\alpha_\Gamma(\varphi_1, \cdots, \varphi_{2k}) = \frac{1}{(2k)!} \sum_{\sigma \in \mathfrak{S}_{2k}} \operatorname{sgn} \sigma \ \alpha_C(\xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(2k)})$$

where $\varphi_i \in \mathcal{M}_{g,*}$ $(i = 1, \cdots, 2k)$, $\xi_i = \rho_0(\varphi_1 \cdots \varphi_{i-1})\tilde{k}(\varphi_i)$ and $\alpha_C : H_{\mathbb{Q}}^{\otimes 6k} \rightarrow \mathbb{Q}$ is the homomorphism given in (5), §3.

Definition 2.2. For any $2k$ elements $\varphi_i \in \mathcal{M}_g$ $(i = 1, \cdots, 2k)$, choose $\tilde{\varphi}_i \in \mathcal{M}_{g,*}$ such that $\tilde{\varphi}_i$ projects to φ_i under the natural projection $\mathcal{M}_{g,*} \rightarrow \mathcal{M}_g$. Then we set

$$\beta_\Gamma(\varphi_1, \cdots, \varphi_{2k}) = \frac{1}{(2k)!} \sum_{\sigma \in \mathfrak{S}_{2k}} \operatorname{sgn} \sigma \ \alpha_C(q(\xi_{\sigma(1)}) \otimes \cdots \otimes q(\xi_{\sigma(2k)}))$$

where $\xi_i = \rho_0(\tilde{\varphi}_1 \cdots \tilde{\varphi}_{i-1})\tilde{k}(\tilde{\varphi}_i)$ and $q : \Lambda^3 H_{\mathbb{Q}} \rightarrow \Lambda^3 H_{\mathbb{Q}}$ is the homomorphism given in (7), §3.

It was proved in the above cited papers that β_Γ is well defined independent of the choices of $\tilde{\varphi}_i$ and also that α_Γ and β_Γ are in fact cocycles. Then by making use of the notion of generalized Mumford-Morita-Miller classes due to Kawazumi [10], the following result was proved in [11].

Theorem 2.3. *The images of the homomorphisms $\Phi_{g,*}$ and Φ_g in* (4) *coincide exactly with the tautological algebras $\mathcal{R}^*(\mathcal{M}_{g,*})$ and $\mathcal{R}^*(\mathcal{M}_g)$ respectively. Moreover, for any element $\Gamma \in \mathcal{G}_{2k}$ we have*

$$\alpha_\Gamma = (-1)^k e_k + e(\textit{lower terms}) \in \mathcal{R}^{2k}(\mathcal{M}_{g,*}).$$

Remark 2.4. For more details concerning the above theorem, in particular explicit forms of the cohomology classes α_Γ and β_Γ, incuding their dependence on the genus g, see our paper [12].

3. Description of Sp-invariant tensors

Let $H = H_1(\Sigma_g; \mathbb{Z})$ and $H_{\mathbb{Q}} = H_1(\Sigma_g; \mathbb{Q})$ as before. $H_{\mathbb{Q}}$ is the fundamental representation of the algebraic group $Sp(2g, \mathbb{Q})$. In this section we prepare several facts concerning Sp-invariant tensors of various Sp-modules related to the mapping class groups. We first recall a few results about invariant tensors of $H_{\mathbb{Q}}^{\otimes 2k}$ and its dual from our earlier papers [22][23].

Choose any symplectic basis x_i, y_i $(i = 1, \cdots, g)$ of H. Then as is well known, the element

$$\omega_0 = \sum_{i=1}^{g} (x_i \otimes y_i - y_i \otimes x_i) \in H \otimes H$$

does not depend on the choice of the symplectic basis and is invariant under the natural action of $Sp(2g, \mathbb{Z})$. Also we consider the skew symmetric bilinear form

$$\mu : H \otimes H \longrightarrow \mathbb{Z}$$

induced by the intersection number which is also $Sp(2g,\mathbb{Z})$-invariant. Now it is a classical result of Weyl that any invariant tensor of $H_{\mathbb{Q}}^{\otimes 2k}$ or its dual $\mathrm{Hom}(H_{\mathbb{Q}}^{\otimes 2k},\mathbb{Q})$, namely any element of $(H_{\mathbb{Q}}^{\otimes 2k})^{Sp}$ or $\mathrm{Hom}(H_{\mathbb{Q}}^{\otimes 2k},\mathbb{Q})^{Sp}$ can be described by iterated applications of the symplectic class $\omega_0 \in (H_{\mathbb{Q}} \otimes H_{\mathbb{Q}})^{Sp}$ or the intersection pairing $\mu \in \mathrm{Hom}(H_{\mathbb{Q}}^{\otimes 2},\mathbb{Q})^{Sp}$.

The way of the above applications can be described by a *linear chord diagram* C with $2k$ vertices which is a decomposition of the set of labeled vertices $\{1,2,\cdots,2k\}$ into pairs $\{(i_1,j_1),(i_2,j_2),\cdots,(i_k,j_k)\}$ such that $i_1<j_1, i_2<j_2,\cdots,i_k<j_k$ (cf. [1]). We also consider C as a graph with k edges by connecting two vertices in each pair (i_s,j_s) by an edge. We define $\operatorname{sgn} C$ by

$$\operatorname{sgn} C = \operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & 2k-1 & 2k \\ i_1 & j_1 & \cdots & i_k & j_k \end{pmatrix}.$$

It is easy to see that there are exactly $(2k-1)!!$ linear chord diagrams with $2k$ vertices. Let us write

$$\mathcal{D}^\ell(2k) = \{C_i; i=1,\cdots,(2k-1)!!\}$$

for the set of all linear chord diagrams with $2k$ vertices. For each element $C \in \mathcal{D}^\ell(2k)$, let

$$a_C \in (H_{\mathbb{Q}}^{\otimes 2k})^{Sp}$$

be the invariant tensor defined by permuting the tensor product $(\omega_0)^{\otimes k}$ in such a way that the s-th part $(\omega_0)_s$ goes to $(H_{\mathbb{Q}})_{i_s} \otimes (H_{\mathbb{Q}})_{j_s}$, where $(H_{\mathbb{Q}})_i$ denotes the i-th component of $H_{\mathbb{Q}}^{\otimes 2k}$, and multiplied by the factor $\operatorname{sgn} C$. We also consider the dual element

$$\alpha_C \in \mathrm{Hom}(H_{\mathbb{Q}}^{\otimes 2k},\mathbb{Q})^{Sp}$$

which is defined by applying the intersection pairing μ on each two components corresponding to pairs (i_s,j_s) of C and multiplied by $\operatorname{sgn} C$. Namely we set

$$\alpha_C(u_1 \otimes \cdots \otimes u_{2k}) = \operatorname{sgn} C \prod_{s=1}^{k} u_{i_s} \cdot u_{j_s} \quad (u_i \in H_{\mathbb{Q}}). \tag{5}$$

Next we consider Sp-invariant tensors of $\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}})$ and its dual. We have a natural projection

$$p : H_{\mathbb{Q}}^{\otimes 6k} \longrightarrow \Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}})$$

as well as injection

$$i : \Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}) \longrightarrow H_{\mathbb{Q}}^{\otimes 6k}$$

given by

$$p(u_1 \otimes \cdots \otimes u_{6k}) = (u_1 \wedge u_2 \wedge u_3) \wedge \cdots \wedge (u_{6k-2} \wedge u_{6k-1} \wedge u_{6k})$$

and

$$i(\xi_1 \wedge \cdots \wedge \xi_{2k}) = \sum_{\sigma \in \mathfrak{S}_{2k}} \operatorname{sgn}\sigma \ \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(2k)}$$

where $\xi_i \in \Lambda^3 H_{\mathbb{Q}}$ and the inclusion $\Lambda^3 H_{\mathbb{Q}} \subset H_{\mathbb{Q}}^{\otimes 3}$ is defined similarly. Hence for each $C \in \mathcal{D}^\ell(6k)$ we have the associated Sp-invariant tensors

$$p(a_C) \in (\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}))^{Sp}, \quad i^*(\alpha_C) \in \mathrm{Hom}(\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}),\mathbb{Q})^{Sp}.$$

We make the following definition.

Definition 3.1. For each $C \in \mathcal{D}^\ell(6k)$, we define Γ_C to be the trivalent graph with $2k$ vertices which is obtained from C, regarded as a disjoint union of $3k$ edges, by joining the three vertices in each of the sets $\{1,2,3\}, \{4,5,6\}, \cdots, \{6k-2, 6k-1, 6k\}$ to a single point.

Then it is easy to see that both $p(a_C)$ and $i^*(\alpha_C)$ depend only on Γ_C. Moreover it is clear that any trivalent graph Γ with $2k$ vertices can be lifted to a linear chord diagram C in $\mathcal{D}^\ell(6k)$ such that $\Gamma_C \cong \Gamma$. Hence we can define Sp-invariant tensors

$$a_\Gamma \in (\Lambda^{2k}(\Lambda^3 H_\mathbb{Q}))^{Sp} \quad \alpha_\Gamma \in \mathrm{Hom}(\Lambda^{2k}(\Lambda^3 H_\mathbb{Q}), \mathbb{Q})^{Sp}$$

for any Γ by setting $a_\Gamma = p(a_C)$ and $\alpha_\Gamma = \frac{1}{(2k)!} i^*(\alpha_C)$, respectively, where C is any lift of Γ. In §2, we already used the same symbol α_Γ for the cocycle of $\mathcal{M}_{g,*}$ associated to the element α_Γ defined above. However, it should not yield any confusion because hereafter we use this symbol only with the former meaning.

Let $C : \Lambda^3 H \to H$ be the contraction map defined by

$$C(u \wedge v \wedge w) = 2\{(u \cdot v)w + (v \cdot w)u + (w \cdot u)v\} \quad (u, v, w \in H).$$

As before we consider H as an $Sp(2g, \mathbb{Z})$-submodule of $\Lambda^3 H$ and we denote by U the quotient $\Lambda^3 H/H$. It is easy to see that the composition $\mathrm{Ker}\, C \otimes \mathbb{Q} \to \Lambda^3 H_\mathbb{Q} \to U_\mathbb{Q}$ is an isomorphism and we identify $\mathrm{Ker}\, C \otimes \mathbb{Q}$ with $U_\mathbb{Q}$. Then we have a decomposition

$$\Lambda^3 H_\mathbb{Q} = U_\mathbb{Q} \oplus H_\mathbb{Q} \tag{6}$$

which in fact gives the irreducible decomposition of the $Sp(2g, \mathbb{Q})$-module $\Lambda^3 H_\mathbb{Q}$. Explicitly the decomposition is given as follows. Let $q : \Lambda^3 H_\mathbb{Q} \to \Lambda^3 H_\mathbb{Q}$ be the map defined by

$$q(\xi) = \xi - \frac{1}{2g-2} C\xi \wedge \omega_0. \tag{7}$$

Then $C \circ q = 0$ so that $\mathrm{Im}\, q \subset \mathrm{Ker}\, C \otimes \mathbb{Q}$ and we have

$$\Lambda^3 H_\mathbb{Q} \ni \xi \longmapsto (q(\xi), \widetilde{C}\xi) \in U_\mathbb{Q} \oplus H_\mathbb{Q} = \Lambda^3 H_\mathbb{Q}$$

where

$$\widetilde{C}\xi = \frac{1}{2g-2} C\xi \wedge \omega_0 \quad (\xi \in \Lambda^3 H_\mathbb{Q}). \tag{8}$$

The above decomposition (6) induces a canonical decomposition

$$\begin{aligned} &\Lambda^{2k}(\Lambda^3 H_\mathbb{Q}) \\ &\cong \Lambda^{2k} U_\mathbb{Q} \oplus (\Lambda^{2k-1} U_\mathbb{Q} \otimes H_\mathbb{Q}) \oplus \cdots \oplus (U_\mathbb{Q} \otimes \Lambda^{2k-1} H_\mathbb{Q}) \oplus \Lambda^{2k} H_\mathbb{Q}. \end{aligned} \tag{9}$$

Hence any element $\xi \in \Lambda^{2k}(\Lambda^3 H_\mathbb{Q})$ can be expressed uniquely as

$$\xi = \xi^{2k,0} + \xi^{2k-1,1} + \cdots + \xi^{1,2k-1} + \xi^{0,2k}$$

where $\xi^{2k-p,p} \in \Lambda^{2k-p} U_\mathbb{Q} \otimes \Lambda^p H_\mathbb{Q}$. If $\xi = \xi_1 \wedge \cdots \wedge \xi_{2k}$ $(\xi_i \in \Lambda^3 H_\mathbb{Q})$, then

$$\xi_i = q(\xi_i) + \widetilde{C}\xi_i$$

so that

$$\xi^{2k-p,p} = \sum_{i_1 < \cdots < i_p} q(\xi_1) \wedge \cdots \wedge \widetilde{C}\xi_{i_1} \wedge \cdots \wedge \widetilde{C}\xi_{i_p} \wedge \cdots \wedge q(\xi_{2k}). \tag{10}$$

In particular

$$\xi^{2k,0} = q(\xi_1) \wedge \cdots \wedge q(\xi_{2k}), \quad \xi^{0,2k} = \widetilde{C}\xi_1 \wedge \cdots \wedge \widetilde{C}\xi_{2k}.$$

Let Γ be a trivalent graph with $2k$ vertices. Then the associated invariant tensor

$$a_\Gamma \in (\Lambda^{2k}(\Lambda^3 H_\mathbb{Q}))^{Sp}$$

can be decomposed as

$$a_\Gamma = a_\Gamma^{2k,0} + a_\Gamma^{2k-1,1} + \cdots + a_\Gamma^{1,2k-1} + a_\Gamma^{0,2k} \tag{11}$$

where $a_\Gamma^{2k-p,p} \in (\Lambda^{2k-p} U_\mathbb{Q} \otimes \Lambda^p H_\mathbb{Q})^{Sp}$. In particular, we simply write

$$\bar{a}_\Gamma = a_\Gamma^{2k,0} \in (\Lambda^{2k} U_\mathbb{Q})^{Sp}.$$

To compute the above decompositions, we prepare a few facts. First, in the above terminology we define another series of elements $\xi^{(p)}$ $(p = 0, 1, \cdots, 2k)$ by

$$\xi^{(p)} = \sum_{i_1 < \cdots < i_p} \xi_1 \wedge \cdots \wedge \widetilde{C}\xi_{i_1} \wedge \cdots \wedge \widetilde{C}\xi_{i_p} \wedge \cdots \wedge \xi_{2k}. \tag{12}$$

In particular

$$\xi^{(0)} = \xi_1 \wedge \cdots \wedge \xi_{2k} = \xi, \quad \xi^{(2k)} = \widetilde{C}\xi_1 \wedge \cdots \wedge \widetilde{C}\xi_{2k} = \xi^{0,2k}.$$

If we apply the above to the invariant tensor $a_\Gamma \in (\Lambda^{2k}(\Lambda^3 H_\mathbb{Q}))^{Sp}$, we obtain the associated series of elements $a_\Gamma^{(p)}$. We introduce these elements because they are more convenient for computations than $\xi^{2k-p,p}$.

Lemma 3.2. *Two series of elements $\xi^{2k-p,p}$ and $\xi^{(p)}$ for $p = 0, \cdots, 2k$ are related to each other by*

$$\xi^{(p)} = \sum_{s=0}^{2k-p} \binom{p+s}{p} \xi^{2k-p-s,p+s}$$

$$\xi^{2k-p,p} = \sum_{s=0}^{2k-p} (-1)^s \binom{p+s}{p} \xi^{(p+s)}.$$

In particular we have

$$\bar{\xi} = \xi^{2k,0} = \xi - \xi^{(1)} + \xi^{(2)} - \cdots + \xi^{(2k)}.$$

Proof. If we put $\xi_i = q(\xi_i) + \widetilde{C}\xi_i$ in the defining equation (12) of $\xi^{(p)}$ and compare it with the defining equation (10) of $\xi^{2k-p,p}$, we obtain the first equation. Similarly, by substituting $q(\xi_i) = \xi_i - \widetilde{C}\xi_i$ in (10) and comparing it with (12), we obtain the second equation. □

Let Γ be a trivalent graph with $2k$ vertices and let a_Γ be the associated Sp-invariant tensor of $\Lambda^{2k}(\Lambda^3 H_\mathbb{Q})$. We would like to describe each component

$$a_\Gamma^{2k-p,p} \in (\Lambda^{2k-p} U_\mathbb{Q} \otimes \Lambda^p H_\mathbb{Q})^{Sp}$$

of a_Γ explicitly. By virtue of Lemma 3.2, we have only to obtain formulas for the elements

$$a_\Gamma^{(p)} \quad (p = 0, 1, \cdots, 2k).$$

Let i, j be two indices with $1 \leq i < j \leq 2k$ and let $p_{ij} : H_{\mathbb{Q}}^{\otimes 2k} \to H_{\mathbb{Q}}^{\otimes 2k}$ be the map defined by first taking the contraction of i-th and j-th entries by the intersection pairing μ and then putting ω_0 there. More precisely

$$p_{ij}(u_1 \otimes \cdots \otimes u_{2k}) = (u_i \cdot u_j) \sum_{s=1}^{g} \{ u_1 \otimes \cdots \otimes x_s \otimes \cdots \otimes y_s \otimes \cdots \otimes u_{2k} \\ - u_1 \otimes \cdots \otimes y_s \otimes \cdots \otimes x_s \otimes \cdots \otimes u_{2k} \}.$$

Lemma 3.3. *Let $C \in \mathcal{D}^\ell(2k)$ be any linear chord diagram. Then for any two indices i, j with $1 \leq i < j \leq 2k$, we have*

$$p_{ij}(a_C) = \begin{cases} 2g\, a_C & \{i,j\} \in C \\ -a_{C'} & \{i,j\} \notin C. \end{cases}$$

where C' is the linear chord diagram defined as follows. Let j', i' be indices such that $\{i, j'\}, \{i', j\} \in C$. Then

$$C' = C \setminus \{\{i,j'\}, \{i',j\}\} \cup \{\{i,j\}, \{i',j'\}\}.$$

Proof. If $\{i,j\} \in C$, then the assertion is clear because $\mu(\omega_0) = 2g$. Suppose that $\{i,j\} \notin C$. Then there are several cases to be considered according to the order of size of the four indices i, j', i', j. However it is easy to see that the computation is essentially the same for each of the two cases: $j' < i'$ or $i' < j'$. Hence it is enough to prove the assertion for the following two typical cases. Namely we take $k = 2, i = 1, j = 4$ and consider the two elements

$$a_{C_1}, a_{C_2} \in H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}$$

where $C_1 = \{\{1,2\}, \{3,4\}\}$ $(j' = 2, i' = 3)$ and $C_2 = \{\{1,3\}, \{2,4\}\}$ $(j' = 3, i' = 2)$. Direct computation shows that

$$p_{14}(a_{C_1}) = -a_{C_3}, \quad p_{14}(a_{C_2}) = -a_{C_3}$$

where $C_3 = \{\{1,4\}, \{2,3\}\}$. This completes the proof. □

Example 3.4. It is clear from the definition that if the indices of finitely many mappings $p_{i_s j_s}$ are disjoint, then they are mutually commutative so that we can compute the effect of the product $\Pi_s p_{i_s j_s}$ on any element by applying Lemma 3.3 successively. For example, in the above notation we have

$$p_{14} p_{23}(a_{C_1}) = p_{14}(-a_{C_3}) = -2g\, a_{C_3}.$$

Lemma 3.5. *Let $C \wedge \omega_0 : \Lambda^3 H \to \Lambda^3 H$ be the composition of the projection $p : \Lambda^3 H \to H$ with the inclusion $H \subset \Lambda^3 H$. We define a linear map $C \otimes \omega_0 : H^{\otimes 3} \longrightarrow H^{\otimes 3}$ by setting*

$$u \otimes v \otimes w \longmapsto (u \cdot v)\omega_0 \otimes w + (v \cdot w) u \otimes \omega_0 + (u \cdot w)\omega_0^{(1,3)} \otimes v^{(2)}$$

where the symbol $\omega_0^{(1,3)} \otimes v^{(2)}$ means that we put ω_0 on the first and the third place of $H^{\otimes 3}$ while v is set on the second place. Then the following diagram is commutative

$$\begin{array}{ccc} H^{\otimes 3} & \xrightarrow{C \otimes \omega_0} & H^{\otimes 3} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \Lambda^3 H & \xrightarrow[C \wedge \omega_0]{} & \Lambda^3 H \end{array}$$

where $p : H^{\otimes 3} \rightarrow \Lambda^3 H$ *is the natural projection.*

Proof. Direct computation using the fact $p(\omega_0) = 2\,\omega_0$ implies the result. □

Keeping in mind the above two lemmas, we introduce certain operation on trivalent graphs as follows.

Let Γ be a trivalent graph and let V_Γ denote the set of vertices of Γ. For each vertex $v \in V_\Gamma$, let $E(v)$ be the set of all *germs of edges* starting from v. Here if τ is an edge starting from the vertex v and ending at another vetex v' (which may be the same as v), then the corresponding germ of edge is the part of τ from v to, say its one third point. Thus $E(v)$ has three elements for any v. Now let $\mathcal{P}(v)$ be the set of *pairs* of germs of edges starting from v, namely

$$\mathcal{P}(v) = \{\{\tau, \tau'\}; \tau, \tau' \in E(v), \tau \neq \tau'\}.$$

$\mathcal{P}(v)$ has also three elements for any v.

Let Γ be a trivalent graph and Let $F \subset V_\Gamma$ be a subset. We denote by $\mathcal{P}(F)$ the set of all correspondences

$$F \ni v \longmapsto \varpi(v) \in \mathcal{P}(v).$$

For each element $\varpi \in \mathcal{P}(F)$ as above, we define the associated graph Γ_ϖ as follows. At each vertex $v \in F$, we are given two germs of edges $\varpi(v) = \{\tau, \tau'\} \in \mathcal{P}(v)$. Let p, q be the endpoints (other than v) of τ and τ', respectively. In a neighborhood of v, we first cut Γ at these two points and connect τ and τ' to make a loop (with endpoint v) and at the same time we also connect the complements of τ, τ' in the corresponding edges to make a new edge.

Applying the above operation at each point $v \in F$, we obtain a new graph Γ_ϖ whose set of vertices is the same as that of Γ. Γ_ϖ is also a trivalent graph with possibly finitely many disjoint circles (note that in the above argument if τ and τ' are contained in a single loop, then after the above operation we are left with the same loop plus an extra disjoint circle). The operations $\Gamma \mapsto \Gamma_\varpi$ for various ϖ may be called the *graphic contractions*. Let $\overline{\Gamma}_\varpi$ be the trivalent graph obtained from Γ_ϖ by removing all the disjoint circles and let $\ell(\varpi)$ be the number of these circles.

Proposition 3.6. *Let* Γ *be a trivalent graph with* $2k$ *vertices and let* a_Γ *be the associated Sp-invariant tensor of* $\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}})$. *Then for each* p *with* $0 \leq p \leq 2k$, *we have*

$$a_\Gamma^{(p)} = \frac{1}{(2g-2)^p} \sum_{F \subset V_\Gamma} \sum_{\varpi \in \mathcal{P}(F)} (-1)^p (-2g)^{\ell(\varpi)}\, a_{\overline{\Gamma}_\varpi}$$

where F *runs through all subsets of* V_Γ *consisting of* p *elements (so that the above expression is the sum of* $3^p \binom{2k}{p}$ *terms). Each term* $a_\Gamma^{2k-p,p}$ *of the decomposition of* a_Γ *in* (11) *is given by combining the above formula with Lemma 3.2. In particular*

$$\bar{a}_\Gamma = a_\Gamma^{(0)} - a_\Gamma^{(1)} + a_\Gamma^{(2)} - \cdots + a_\Gamma^{(2k)}.$$

Proof. Lemma 3.3 and Lemma 3.5 show that the algebraic definition of $a_\Gamma^{(p)}$ (see (12)) can be realized geometrically by the graphic contractions $\Gamma \mapsto \Gamma_\varpi$. The result follows. □

4. Degeneration of Sp-invariant tensors

We know that

$$\dim(H_{\mathbb{Q}}^{\otimes 2k})^{Sp} = (2k-1)!! \quad (g \geq k)$$

and a_C ($C \in \mathcal{D}^\ell(2k)$) is a basis of it (see [23]). Similar statement holds for the dual space $\mathrm{Hom}(H_{\mathbb{Q}}^{\otimes 2k}, \mathbb{Q})$. One consequence of this fact together with the Poincaré duality

$$H_{\mathbb{Q}}^{\otimes 2k} \cong \mathrm{Hom}(H_{\mathbb{Q}}^{\otimes 2k}, \mathbb{Q})$$

which is induced by $H_{\mathbb{Q}} \cong H_{\mathbb{Q}}^*$ is that the matrix

$$\Big(\alpha_C(a_{C'})\Big)_{C,C' \in \mathcal{D}^\ell(2k)}$$

is non-singular for $g \geq k$. Here the entries of the above matrix is given by

$$\alpha_C(a_{C'}) = (-1)^{k-r}(2g)^r \tag{13}$$

where r denotes the number of connected components of the graph $C \cup C'$ (see [23][12]).

Proposition 4.1. *If $g = k-1$, then*

$$\dim(H_{\mathbb{Q}}^{\otimes 2k})^{Sp} = (2k-1)!! - 1$$

and the unique relation is

$$\sum_{C \in \mathcal{D}^\ell(2k)} a_C = 0.$$

Proof. We first prove that the relation above actually holds for $g \leq k-1$. For that it is sufficient to show that

$$\sum_{C \in \mathcal{D}^\ell(2k)} \alpha_{C_0}(a_C) = 2^k g(g-1)\cdots(g-k+1) \tag{14}$$

for any fixed $C_0 \in \mathcal{D}^\ell(2k)$. It is clear that the above sum is independent of the choice of C_0. Therefore we may assume that

$$C_0 = \{\{1,2\},\{3,4\},\cdots,\{2k-1,2k\}\}.$$

For each r with $1 \leq r \leq k$, let $p(2k,r)$ denote the number of linear chord diagrams $C \in \mathcal{D}^\ell(2k)$ such that the number of connected components of $C_0 \cup C$ is equal to r. Then by (13) clearly we have

$$\sum_{C \in \mathcal{D}^\ell(2k)} \alpha_{C_0}(a_C) = \sum_{r=1}^{k} (-1)^{k-r} p(2k,r)(2g)^r.$$

Hence to prove (14) it suffices to show that

$$\sum_{r=1}^{k} (-1)^{k-r} p(2k,r)(2g)^r = 2^k g(g-1)\cdots(g-k+1). \tag{15}$$

For each $C \in \mathcal{D}^\ell(2k+2)$, let n_C denote the integer such that $\{1, n_C\} \in C$. Then by considering the cases where $n_C = 2$ and $n_C \neq 2$ (there are $2k$ such cases), it is easy to see that the equality

$$p(2k+2,r) = p(2k,r-1) + 2k\, p(2k,r) \tag{16}$$

holds (here we set $p(2k,0)=p(2k,k+1)=0$). We now prove (15) (and hence (14)) by induction on k. The assertion is clear for $k=1$. Assume that it holds for $1,\cdots,k$ $(k\geq 1)$. Using the induction assumption and (16), we can compute as follows.

$$
\begin{aligned}
&\sum_{r=1}^{k+1}(-1)^{k+1-r}p(2k+2,r)(2g)^r\\
=&\sum_{r=1}^{k+1}(-1)^{k+1-r}\{p(2k,r-1)+2k\,p(2k,r)\}(2g)^r\\
=&\sum_{r=1}^{k}(-1)^{k-r}p(2k,r)(2g)^{r+1}+2k\sum_{r=1}^{k}(-1)^{k+1-r}p(2k,r)(2g)^r\\
=&(2g-2k)2^k g(g-1)\cdots(g-k+1)\\
=&2^{k+1}g(g-1)\cdots(g-k).
\end{aligned}
$$

Thus we proved (14).

Next we prove that $\dim(H_{\mathbb{Q}}^{\otimes 2k})^{Sp}$ is at least $(2k-1)!!-1$ for $g=k-1$ by induction on k. For that we first define a total order in the set $\mathcal{D}^\ell(2k)$ inductively as follows. If $k=1$, $\mathcal{D}^\ell(2)$ has only one element so that it is trivially ordered. Assume that we have defined a total order in $\mathcal{D}^\ell(2k)$ and let

$$C_1<C_2<\cdots<C_{n_k}\quad (n_k=(2k-1)!!)$$

be the corresponding sequence. Then we define an order in $\mathcal{D}^\ell(2k+2)$ as follows. We begin by

$$
\begin{array}{ccccccccc}
\{1,2\}\cup C_1^2 & < & \{1,2\}\cup C_2^2 & < & \cdots & < & \{1,2\}\cup C_{n_k}^2 & < \\
\{1,3\}\cup C_1^3 & < & \{1,3\}\cup C_2^3 & < & \cdots & < & \{1,3\}\cup C_{n_k}^3 & < \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

and end with

$$\{1,2k+2\}\cup C_1^{2k+2}<\{1,2k+2\}\cup C_2^{2k+2}<\cdots<\{1,2k+2\}\cup C_{n_k}^{2k+2}$$

where C_j^m denotes the linear chord diagram with vertices $2,\cdots,\hat{m},\cdots,2k+2$ which corresponds to C_j by the renumbering of indices in order of size where the symbol $\hat{m}$ means we omit m. Next we define a series of elements $u_1^{2k},\cdots,u_{n_k}^{2k}\in H_{\mathbb{Q}}^{\otimes 2k}$ $(g=k-1)$ again inductively as follows. If $k=2$, we set

$$u_1^4=x_1\otimes y_1\otimes x_1\otimes y_1,\ u_2^4=-x_1\otimes x_1\otimes y_1\otimes y_1,\ u_3^4=x_1\otimes x_1\otimes y_1\otimes y_1$$

so that $u_3^4=-u_2^4$. Then the intersection matrix $\left(\alpha_{C_i}(u_j^4)\right)$ is given by

$$\begin{pmatrix}1&0&0\\0&1&-1\\0&-1&1\end{pmatrix}$$

whose rank is $2=3!!-1$. Assume that u_j^{2k-2} $(j=1,\cdots,n_{k-1})$ have been defined. Then we define u_j^{2k} $(j=1,\cdots,n_k)$ as follows. For $1\leq j\leq n_k-n_{k-1}$, let $\{1,f_j\}$ be the first chord in C_j. Then we put $x_{k-1}\otimes y_{k-1}$ on the first and the f_j-th factors of $H_{\mathbb{Q}}^{\otimes 2k}$. We arrange the remaining $k-1$ chords in C_j according to the size of the smaller number of each chord and let $\{s_\ell,t_\ell\}$ $(\ell=1,\cdots,k-1)$ be the ℓ-th chord where $s_\ell<t_\ell$. Then we put $x_\ell\otimes y_\ell$ $(\ell=1,\cdots,k-1)$ on the s_ℓ-th and t_ℓ-th factors

of $H_{\mathbb{Q}}^{\otimes 2k}$. We finally multiply the resulting tensor by $\operatorname{sgn} C_j$ to obtain u_j^{2k}. For $n_k - n_{k-1} + 1 \leq j \leq n_k$, C_j has the form

$$C_j = \{1, 2k\} \cup \overline{C}_{j'}^{2k}$$

where $j' = j - n_k + n_{k-1}$ so that $j' = 1, \cdots, n_{k-1}$ and $\overline{C}_{j'}$ denotes the totally ordered elements for the case with $2k - 2$ vertices. The last two elements are linearly dependent. In fact $C_{n_k} = -C_{n_k - 1}$. Now we claim that the rank of the matrix

$$\left(\alpha_{C_i}(u_j^{2k})\right) \quad (i, j = 1, \cdots, n_k)$$

is equal to $n_k - 1$. If $k = 2$, this was already shown above. For general k, an inductive argument shows that the above matrix has the form

$$\begin{pmatrix} 1 & 0 & \ldots\ldots & 0 & 0 & 0 \\ * & 1 & 0 \quad \ldots & 0 & 0 & 0 \\ * & * & 1 \quad 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ * & & * & 1 & 0 & 0 \\ * & \ldots & \ldots & * & 1 & -1 \\ * & \ldots & \ldots & * & -1 & 1 \end{pmatrix}$$

so that its rank is $n_k - 1$ as required. Thus we see that the elements α_{C_i} $(i = 1, \cdots, n_k - 1)$ are linearly independent. By the obvious duality, this completes the proof. □

Remark 4.2. The dimension formula above is known classically. On the other hand, (the Poincaré dual of) the unique relation above comes from the fact that $(\omega_0^*)^{g+1} = 0 \in \left(\Lambda^{2g+2} H_{\mathbb{Q}}^*\right)^{Sp} = \{0\}$ where $\omega_0^* \in \left(\Lambda^2 H_{\mathbb{Q}}^*\right)^{Sp}$ denotes the symplectic form. See also [16] for a recent result about Sp-invariant tensors in the unstable range.

5. The first relations

In this section, we show that the k-th Mumford-Morita-Miller class e_k can be expressed as a polynomial in the lower classes $e_1, \cdots, e_{k-1}$ for the genus $g = 3k - 1$ for all k. We call these the *first relations* because they are the first of infinitely many series of relations in the tautological algebra $\mathcal{R}^*(\mathcal{M}_g)$ which we are going to prove.

We begin by describing our method of obtaining relations. Recall from §3 that, associated to any linear chord diagram $C \in \mathcal{D}^\ell(6k)$, we have certain Sp-invariant tensors

$$a_C \in (H_{\mathbb{Q}}^{\otimes 6k})^{Sp}, \quad \alpha_C \in \operatorname{Hom}(H_{\mathbb{Q}}^{\otimes 6k}, \mathbb{Q})^{Sp}$$

and passing to the natural quotient (as well as sub) space $\Lambda^{2k}\Lambda^3 H_{\mathbb{Q}}$ we have Sp-invariant tensors

$$a_{\Gamma_C} \in (\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}))^{Sp}, \quad \alpha_{\Gamma_C} \in \operatorname{Hom}(\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}), \mathbb{Q})^{Sp}.$$

By Theorem 2.3 and also by abuse of language (see §2), we can regard the last tensor as an element of the tautological algebra of $\mathcal{M}_{g,*}$. Thus we can write

$$\alpha_{\Gamma_C} \in \mathcal{R}^{2k}(\mathcal{M}_{g,*})$$

where g can be any genus (however we mention that α_{Γ_C} *does* depend on g).

If there is a linear relation

$$\sum_i \rho_i \, a_{C_i} = 0$$

for some genus g, then it induces a relation

$$\sum_i \rho_i \, \alpha_{\Gamma_{C_i}} = 0 \in \mathcal{R}^{2k}(\mathcal{M}_{g,*}).$$

In particular, the linear relation

$$\sum_{C \in \mathcal{D}^\ell(6k)} a_C = 0$$

given in Proposition 4.1, which holds for any $g \leq 3k-1$, yields the relations

$$\sum_{C \in \mathcal{D}^\ell(6k)} \alpha_{\Gamma_C} = 0 \in \mathcal{R}^{2k}(\mathcal{M}_{g,*})$$

for all genera $g \leq 3k-1$. However it is not so easy to identify these relations explicitly because the formula expressing α_Γ as polynomials in $e, e_1, e_2, \cdots$ is complicated (see [12]). Also the above relations hold in $\mathcal{R}^*(\mathcal{M}_{g,*})$ and not in $\mathcal{R}^*(\mathcal{M}_g)$. Although we can apply the integration along the fiber $\mathcal{R}^*(\mathcal{M}_{g,*}) \to \mathcal{R}^{*-2}(\mathcal{M}_g)$ to the above relations, multiplied by any power of e, to obtain many relations in $\mathcal{R}^*(\mathcal{M}_g)$ (see Example 9.1 and Example 9.2 in §9), here we consider more essential operations.

For that we denote the left hand side of the unique relation given in Proposition 4.1 simply by $\tilde{r}_k$. Thus

$$\tilde{r}_k = \sum_{C \in \mathcal{D}^\ell(6k)} a_C \in H_{\mathbb{Q}}^{\otimes 6k}.$$

Also we write $r_k \in \Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}})$ and $\bar{r}_k \in \Lambda^{2k} U_{\mathbb{Q}}$ for the images of $\tilde{r}_k$ under the canonical projections

$$H_{\mathbb{Q}}^{\otimes 6k} \longrightarrow \Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}}) \longrightarrow \Lambda^{2k} U_{\mathbb{Q}}.$$

In case we specify the genus g, we write $\tilde{r}_k(g)$ and $r_k(g)$ for the corresponding elements. In the stable range, namely if $g \geq 3k$, $r_k(g)$ is a non-trivial element. We know by Proposition 4.1 that $r_k(g) = 0$ for $g \leq 3k-1$.

Now for each p with $0 \leq p \leq 2k$, consider the element $r_k^{(p)}$ associated to r_k which was defined in §3 (see (12)). We know by Lemma 3.2 that

$$\bar{r}_k = r_k - r_k^{(1)} + r_k^{(2)} - \cdots + r_k^{(2k)} \tag{17}$$

Proposition 5.1. *For any genus $g \leq 3k-1$, we have relations*

$$r_k^{(p)} = 0 \quad (0 \leq p \leq 2k).$$

In particular $\bar{r}_k = 0$.

Proof. By Proposition 4.1, we already know that $\tilde{r}_k = 0$ for $g \leq 3k-1$. Hence both r_k and $\bar{r}_k$ are zero because they are projections of $\tilde{r}_k$. It follows that, for any p with $0 \leq p \leq 2k$, the component $r_k^{2k-p,p}$ of r_k with respect to the direct sum decomposition (9) vanishes. In view of Lemma 3.2, the claim now follows from this. □

Definition 5.2. For each genus g, let

$$\alpha_k(g) \in \mathbb{Q}[e, e_1, e_2, \cdots, e_k]$$

be the characteristic class corresponding to the element r_k. Namely in the terminology of §2

$$\alpha_k(g) = \sum_{C \in \mathcal{D}^\ell(6k)} \alpha_{\Gamma_C}$$

(see (4) and Theorem 2.3). Similarly we define

$$\alpha_k^{(p)}(g), \bar{\alpha}_k(g) \in \mathbb{Q}[e, e_1, e_2, \cdots, e_k]$$

to be the elements corresponding to $r_k^{(p)}$ and $\bar{r}_k$.

Proposition 5.3. *For any positive integer k and any p with $0 \leq p \leq 2k$, we have the relation*

$$\alpha_k^{(p)}(g) = 0$$

which holds in $\mathcal{R}^{2k}(\mathcal{M}_{g,*})$ for any $g \leq 3k-1$. Moreover the polynomial $\bar{\alpha}_k(g)$ does not contain any term containing the Euler class e. Hence we have the relation

$$\bar{\alpha}_k(g) = 0$$

which holds in $\mathcal{R}^{2k}(\mathcal{M}_g)$ for any $g \leq 3k-1$.

Proof. This follows immediately from Proposition 5.1 except for the claim that $\bar{\alpha}_k(g)$ does not contain terms with e so that the induced relation holds in $\mathcal{R}^{2k}(\mathcal{M}_g)$ rather than in $\mathcal{R}^{2k}(\mathcal{M}_{g,*})$. This is essentially a consequence of the Poincaré duality. More precisely, the usual Poincaré duality $H_\mathbb{Q} \cong H_\mathbb{Q}^*$ induces a canonical isomorphism

$$\Lambda^3 H_\mathbb{Q} \cong \Lambda^3 H_\mathbb{Q}^* \tag{18}$$

under which $U_\mathbb{Q} \subset \Lambda^3 H_\mathbb{Q}$ goes to $U_\mathbb{Q}^*$. In particular, the Poincaré dual of any element in $U_\mathbb{Q}$ annihilates $H_\mathbb{Q} \subset \Lambda^3 H_\mathbb{Q}$. Now (18) induces a canonical isomorphism

$$\left(\Lambda^{2k}(\Lambda^3 H_\mathbb{Q})\right)^{Sp} \cong \mathrm{Hom}(\Lambda^{2k}(\Lambda^3 H_\mathbb{Q}), \mathbb{Q})^{Sp}$$

under which the element a_Γ corresponds to α_Γ (up to non-zero scalar). Then it is clear that the Poincaré dual of any element in $(\Lambda^{2k} U_\mathbb{Q})^{Sp}$ takes the value 0 on any tensor

$$\xi_1 \wedge \cdots \wedge \xi_{2k} \in \Lambda^{2k}(\Lambda^3 H_\mathbb{Q})$$

whenever at least one component ξ_i is contained in $H_\mathbb{Q} \subset \Lambda^3 H_\mathbb{Q}$. It follows that the corresponding characteristic class in $H^{2k}(\mathcal{M}_{g,*}; \mathbb{Q})$ actually comes from $H^{2k}(\mathcal{M}_g; \mathbb{Q})$ under the natural injection $H^*(\mathcal{M}_g; \mathbb{Q}) \to H^*(\mathcal{M}_{g,*}; \mathbb{Q})$. Hence it does not contain the Euler class e. Since $a_k(g)$ is contained in $(\Lambda^{2k} U_\mathbb{Q})^{Sp}$, the claim follows.

In fact, in the terminology of Definition 2.2 we have

$$\bar{\alpha}_k(g) = \sum_{C \in \mathcal{D}^\ell(6k)} \beta_{\Gamma_C}.$$

□

Thus we see that the single relation $\tilde{r}_k = 0$ in the unstable range gives rise to many associated relations in the tautological algebras. Our main task is then to write down these relations explicitly.

First we consider the case $g = 3k - 1$. Proposition 4.1 shows that $\tilde{r}_k = 0$ is the *unique* relation in $(H_{\mathbb{Q}}^{\otimes 6k})^{Sp}$ in this case. On the other hand, it is easy to check that the induced relation $\bar{r}_k = 0$ is also non-trivial. Hence this relation is the same as $r_k = 0$ and all the other relations $r_k^{(p)}$ $(0 < p \leq 2k)$ are trivial. In other words, the unique relation occurs *inside* $\Lambda^{2k} U_{\mathbb{Q}}$. Hence the polynomial $\alpha_k(3k-1)$ does not contain e by Proposition 5.3.

In view of Theorem 2.3, we have only to enumerate the numbers of vertices of the connected components of trivalent graphs Γ_C where C runs through the set $\mathcal{D}^\ell(6k)$ of all linear chord diagrams with $6k$ vertices. To analize this, we make the following definitions.

Definition 5.4. Let Γ be a trivalent graph with $2k$ vertices. We define the *connected type* of Γ to be the partition $k = i_1 j_1 + \cdots + i_s j_s$ of k with $i_1 < \cdots < i_s$ and $j_\ell > 0$, where j_ℓ denotes the number of connected components of Γ which have $2i_\ell$ vertices. In particular, all the *connected* trivalent graphs with $2k$ vertices have the same connected type corresponding to the trivial partition $k = k$.

Definition 5.5. For each natural number k, we define b_k to be the number of linear chord diagrams $C \in \mathcal{D}^\ell(6k)$ such that the associated trivalent graph Γ_C is *connected*.

There are $(6k-1)!!$ elements in $\mathcal{D}^\ell(6k)$ so that we can write

$$(6k-1)!! = b_k + \text{decomposable terms}$$

where by *decomposable terms* we mean those terms which correspond to linear chord diagrams C with *disconnected* Γ_C. For the description of these terms, it is convenient to introduce another number b'_k defined by

$$b'_k = \frac{b_k}{(2k-1)!!\,k!}$$

because we then have the following formula which should be considered as the generating function for the enumeration of our trivalent graphs Γ_C according to their connected types.

Proposition 5.6.

$$\exp\Big(\sum_{k=1}^{\infty} b'_k\, t^k\Big) = \sum_{k=0}^{\infty} \frac{(6k-1)!!}{(2k-1)!!\,k!}\, t^k.$$

Proof. We classify $(6k-1)!!$ linear chord diagrams $C \in \mathcal{D}^\ell(6k)$ according to the connected types of the associated trivalent graphs Γ_C as follows. First of all, the number of C with connected Γ_C is b_k, just by definition. For each partition π of k given by

$$k = i_1 j_1 + \cdots + i_s j_s$$

with $i_1 < \cdots < i_s$ and $j_\ell \geq 1$, let b_π denote the number of those C whose associated trivalent graph Γ_C has the connected type corresponding to π. Then we have

$$\begin{aligned}
b_\pi =& \frac{\binom{2k}{2i_1}\binom{2k-2i_1}{2i_1}\cdots\binom{2k-2(j_1-1)i_1}{2i_1}}{j_1!}\, b_{i_1}^{j_1} \times \\
& \frac{\binom{2k-2j_1i_1}{2i_2}\binom{2k-2j_1i_1-2i_2}{2i_2}\cdots\binom{2k-2j_1i_1-2(j_2-1)i_2}{2i_2}}{j_2!}\, b_{i_2}^{j_2} \times\cdots\times \\
& \frac{\binom{2j_si_s}{2i_s}\binom{2j_si_s-2i_s}{2i_s}\cdots\binom{4i_s}{2i_s}\binom{2i_s}{2i_s}}{j_s!}\, b_{i_s}^{j_s}.
\end{aligned}$$

If π is the trivial partition $k = k$, then of course $b_\pi = b_k$ and we have

$$(6k-1)!! = \sum_\pi b_\pi \tag{19}$$

where π runs through all partitions of k. By the definition of the numbers b'_k, we obtain

$$\begin{aligned}
& b_\pi \\
=& \binom{2k}{2i_1}\cdots\binom{2k-2(j_1-1)i_1}{2i_1} \times \binom{2k-2j_1i_1}{2i_2}\cdots\binom{2k-2j_1i_1-2(j_2-1)i_2}{2i_2} \times\cdots\times \\
& \binom{2j_si_s}{2i_s}\cdots\binom{2i_s}{2i_s} \times \Big((2i_1-1)!!\, i_1!\Big)^{j_1}\Big((2i_2-1)!!\, i_2!\Big)^{j_2}\cdots\Big((2i_s-1)!!\, i_s!\Big)^{j_s} \times \\
& \frac{1}{j_1!}\,(b'_{i_1})^{j_1}\cdots\frac{1}{j_s!}\,(b'_{i_s})^{j_s} \\
=& \frac{(2k)!}{\big((2i_1)!\big)^{j_1}\cdots\big((2i_s)!\big)^{j_s}}\Big((2i_1-1)!!\, i_1!\Big)^{j_1}\cdots\Big((2i_s-1)!!\, i_s!\Big)^{j_s}\,\frac{1}{j_1!}\,(b'_{i_1})^{j_1}\cdots\frac{1}{j_s!}\,(b'_{i_s})^{j_s}.
\end{aligned}$$

Since $(2\ell)! = (2\ell-1)!!\, 2^\ell\, \ell!$ for any ℓ, we have

$$\begin{aligned}
& \frac{(2k)!}{\big((2i_1)!\big)^{j_1}\cdots\big((2i_s)!\big)^{j_s}}\Big((2i_1-1)!!\, i_1!\Big)^{j_1}\cdots\Big((2i_s-1)!!\, i_s!\Big)^{j_s} \\
=& \frac{(2k)!}{2^{j_1i_1+\cdots+j_si_s}} = (2k-1)!!\, k!.
\end{aligned}$$

Hence

$$b_\pi = (2k-1)!!\, k!\, \frac{1}{j_1!}\,(b'_{i_1})^{j_1}\cdots\frac{1}{j_s!}\,(b'_{i_s})^{j_s}$$

and we can write

$$\frac{(6k-1)!!}{(2k-1)!!\, k!} = \sum_\pi \frac{1}{j_1!}\,(b'_{i_1})^{j_1}\cdots\frac{1}{j_s!}\,(b'_{i_s})^{j_s}.$$

On the other hand, the right hand side of the above expression is precisely equal to the coefficient of t^k of the formal power series expansion of

$$\exp\Big(\sum_{k=1}^\infty b'_k\, t^k\Big).$$

This completes the proof. $\square$

By virtue of the above proposition, the numbers b_k, b'_k can be determined recursively. Here are the first four values of them:

$$\begin{aligned} &b_1 = 15, \quad b_2 = 9720, \quad b_3 = 32221800, \quad b_4 = 298929657600, \cdots \\ &b'_1 = 15, \quad b'_2 = 1620, \quad b'_3 = 358020, \qquad b'_4 = 118622880, \cdots \end{aligned}$$

Theorem 5.7. *In the rational cohomology group of the moduli spaces* $\mathbf{M}_g$*, we have the relation*

$$\exp\Big(\sum_{k=1}^{\infty} b'_k\, e_k\, t^k\Big) = 0.$$

More precisely, for any k *the coefficient of* t^k*, in the formal power series expansion of the left hand side of the above expression, vanishes as an element of* $H^{2k}(\mathbf{M}_{3k-1};\mathbb{Q})$*. Namely*

$$\sum_{\mathbf{i},\mathbf{j}} \frac{1}{j_1!}\,(b'_{i_1})^{j_1} \cdots \frac{1}{j_s!}\,(b'_{i_s})^{j_s}\;\; e_{i_1}^{j_1} \cdots e_{i_s}^{j_s} = 0. \tag{20}$$

Here the summation is taken over all multi-indices $\mathbf{i} = (i_1, \cdots, i_s)$, $\mathbf{j} = (j_1, \cdots, j_s)$ *with all entries positive integers such that* $i_1 < \cdots < i_s$ *and* $i_1 j_1 + \cdots + i_s j_s = k$*.*

Proof. This follows from Proposition 5.3, Proposition 5.6 and Theorem 2.3. Here the sign $(-1)^k$ in the leading term of α_Γ in Theorem 2.3 causes no problem because $\Pi_i(-1)^{i_1 j_1} \cdots (-1)^{i_s j_s} = (-1)^k$ whenever $i_1 j_1 + \cdots + i_s j_s = k$. □

Corollary 5.8. *For any* k *the class* $e_k \in H^{2k}(\mathbf{M}_{3k-1};\mathbb{Q})$ *can be expressed as a polynomial in* $e_1, \cdots, e_{k-1}$*.*

Proof. This is because the leading term of the coefficient of t^k above is $b'_k e_k$ which is non-zero. □

Example 5.9. Here are explicit first relations for low values of k where we normalize them so that the coefficients are all integers such that the g.c.d. is 1.

$$\begin{aligned} &k = 1, \quad g = 2, \quad e_1 = 0 \\ &k = 2, \quad g = 5, \quad 72\, e_2 + 5\, e_1^2 = 0 \\ &k = 3, \quad g = 8, \quad 15912\, e_3 + 1080\, e_1 e_2 + 25\, e_1^3 = 0 \\ &k = 4, \quad g = 11, \quad 7029504\, e_4 + 318240\, e_1 e_3 + 77760\, e_2^2 + 10800\, e_1^2 e_2 + 125\, e_1^4 = 0. \end{aligned}$$

For later use, we make the following definition.

Definition 5.10. We denote simply by a_k the number $(6k-1)!!$, which is the number of linear chord diagrams with $6k$ vertices. It plays a fundamental role in this paper. We also denote the polynomial $(-1)^k \alpha_k(g)$ for $g = 3k-1$ simply by

$$\alpha_k \in \mathbb{Q}[e_1, e_2, \cdots, e_k].$$

Namely, it is the polynomial obtained by replacing b_π in (19) by $b_\pi e_{i_1}^{j_1} \cdots e_{i_s}^{j_s}$. In other words α_k is equal to the left hand side of (16) multiplied by $(2k-1)!!k!$. Here we put the sign $(-1)^k$ in order to eliminate the sign in later computations. Thus the first four polynomials α_k are given by

$$\begin{aligned}
\alpha_1 =& 15\, e_1\\
\alpha_2 =& 9720\, e_2 + 675\, e_1^2\\
\alpha_3 =& 32221800\, e_3 + 2187000\, e_1e_2 + 50625\, e_1^3\\
\alpha_4 =& 298929657600\, e_4 + 13533156000\, e_1e_3 + 3306744000\, e_2^2\\
& + 459270000\, e_1^2e_2 + 5315625\, e_1^4
\end{aligned}$$

and the sum of all the coefficients of α_k is equal to the number $a_k = (6k-1)!!$.

Remark 5.11. The above result essentially coincides with that of Faber given in [5] which was obtained earlier by a completely different method. Here we describe similarity as well as difference between the two approaches. He defines rational numbers ρ_k recursively by the equation

$$\exp\Big(-\sum_{i=1}^{\infty}\rho_i\, t^i\Big) = \sum_{k=0}^{\infty}\frac{(6k)!}{(2k)!(3k)!}\, t^k$$

(compare with Proposition 5.6) and mentions that the coefficient of t^k in the formal power series

$$\exp\Big(\sum_{i=1}^{\infty}\rho_i\, \kappa_i\, t^i\Big)$$

is 0 in $\mathcal{R}^k(\mathbf{M}_{3k-1})$. Since

$$\sum_{k=0}^{\infty}\frac{(6k)!}{(2k)!(3k)!}\, t^k = \sum_{k=0}^{\infty}\frac{(6k-1)!!}{(2k-1)!!k!}\, 2^{2k}\, t^k$$

and $\kappa_i = (-1)^{i+1}e_i$, we can conclude that $\rho_k = -2^{2k}b_k'$ (and hence ρ_k is negative for any k) and that Faber's relation is $(-1)^k2^{2k}$ times that of ours.

At present our method only yields relations in the rational cohomology of the moduli spaces rather than in the Chow algebras. However there are also certain merits of our approach. First we know the precise geometrical meaning of the numbers b_k (or b_k') so that the reason why the relations hold would appear to be natural. Secondly, once we obtain a relation for some genus g_0 by our method, then we can obtain induced relations for all genera $g \leq g_0$. Finally the coefficients of our equalities are all positive. This is because we are using the classes e_i rather than κ_i (though it is simply a matter of signs). These points will play crucial roles when we later prove non-trivialities of deeper relations.

6. The second relations

Recall from §5 that we denote by $\tilde{r}_k$ the sum of all the elements a_C where C runs through the set $\mathcal{D}^\ell(6k)$. Hence we can write the element r_k, which is the projection of $\tilde{r}_k$ to $\Lambda^{2k}(\Lambda^3 H_{\mathbb{Q}})$, by the form

$$r_k = \sum_j n_j a_{\Gamma_j}.$$

Here Γ_j runs through the set of all the isomorphism classes of trivalent graphs with $2k$ vertices and the coefficients n_j are all positive integers. In general, it is very difficult to determine n_j.

If we classify the trivalent graphs appearing in r_k by their connected types, then we can determine the coefficients explicitly and that is nothing but the polynomial α_k defined in §5.

To obtain deeper relations in the tautological algebra $\mathcal{R}^*(\mathcal{M}_g)$, we have to determine the coefficients of $\bar{r}_k$ with respect to connected types of trivalent graphs appearing in it. For that, we begin with the following proposition.

Proposition 6.1. *Let k be a positive integer and set $g_0 = 3k-1$. Then, for each p with $1 \leq p \leq 2k$, the element $r_k^{(p)}$ can be described in the following form*

$$\begin{aligned}
r_k^{(1)} &= \frac{1}{2g-2}(g-g_0)\sum_j n_j^{(1)} a_{\Gamma_j}\\
r_k^{(2)} &= \frac{1}{(2g-2)^2}(g-g_0)(g-g_0+1)\sum_j n_j^{(2)} a_{\Gamma_j}\\
&\vdots\\
r_k^{(p)} &= \frac{1}{(2g-2)^p}(g-g_0)(g-g_0+1)\cdots(g-g_0+p-1)\sum_j n_j^{(p)} a_{\Gamma_j}\\
&\vdots\\
r_k^{(2k)} &= \frac{1}{(2g-2)^{2k}}(g-g_0)(g-g_0+1)\cdots(g-g_0+2k-1)\sum_j n_j^{(2k)} a_{\Gamma_j}.
\end{aligned}$$

Here all the coefficients $n_j^{(p)}$ are non-negative integers and the sum is given by

$$\sum_j n_j^{(p)} = 2^p 3^p \binom{2k}{p}(6k-2p-1)!!.$$

Proof. Roughly speaking, we can prove the claim by applying Proposition 3.6 to the element r_k. In doing so, by virtue of Lemma 3.5 we can use $\tilde{r}_k$ instead of r_k and apply the operation $C \wedge \omega_0$ on various sub-$H_{\mathbb{Q}}^{\otimes 3}$'s inside $H_{\mathbb{Q}}^{\otimes 6k}$. There are $2k$ such subspaces, namely $\{1,2,3\}, \cdots, \{6k-2, 6k-1, 6k\}$ in coordinates. This operation in turn can be explicitly computed by the operations $p_{ij} : H_{\mathbb{Q}}^{\otimes 6k} \to H_{\mathbb{Q}}^{\otimes 6k}$ described in Lemma 3.3.

More precisely, we first consider the case of $r_k^{(1)}$. Then we have $(6k-1)!!$ linear chord diagrams C in $\tilde{r}_k$ and we have to apply $3 \times 2k$ operations p_{ij} on each C. Here $2k$ is the number of *blocks* $\{1,2,3\}, \cdots, \{6k-2, 6k-1, 6k\}$, while 3 is the number of necessary operations p_{ij} in each block, namely p_{12}, p_{13}, p_{23} for the first block $\{1,2,3\}$ for example. In the language of the associated trivalent graph Γ_C, $2k$ is the number of vertices while 3 is the valency at each vertex of Γ_C. Since $\tilde{r}_k$ contains all the linear chord diagrams evenly, by the obvious symmetry we have only to apply the operation p_{12} on $\tilde{r}_k$ and multiply the result by $3 \times 2k = 6k$. Now consider the first chord $\{1, j_C\}$ of each chord diagram C. By Lemma 3.3, if $j_C = 2$, then $p_{12}(a_C) = 2g\, a_C$ and if $j_C \neq 2$, then $p_{12}(a_C) = -a_{C'}$ for certain C'. If we fix j_C, then there are $(6k-3)!!$ linear chord diagrams with the prescribed value j_C. Thus we can write

$$p_{12}(\tilde{r}_k) = \{2g - (6k-2)\} \sum_{C \in \mathcal{D}^\ell(6k)^{(1)}} a_C \tag{21}$$

where

$$\mathcal{D}^\ell(6k)^{(1)} = \{C \in \mathcal{D}^\ell(6k); \{1,2\} \in C\}.$$

The number of elements of the set $\mathcal{D}^\ell(6k)^{(1)}$ is equal to $(6k-3)!!$. Since $6k\{2g-(6k-2)\} = 2\cdot 3(2k)(g-g_0)$, this proves the claim for $r_k^{(1)}$.

Next we consider $r_k^{(2)}$. In this case, we have to apply the operations p_{ij} on two blocks in each chord diagram C. There are $\binom{2k}{2}$ choices of two blocks out of $2k$ and in each choice we have to apply 3^2 operations. By an obvious symmetry again, we have only to apply the operation $p_{12}\otimes p_{45}$ on $\tilde{r}_k$ and multiply the result by $3^2\times\binom{2k}{2}$. We know already $p_{12}(\tilde{r}_k)$ as described in (21). We have to apply p_{45} to it. Then by a similar argument as above, using Lemma 3.3 and this time considering the chord $\{4, j_C\}$ of each chord diagram $C \in \mathcal{D}^\ell(6k)^{(1)}$, we obtain

$$p_{12}\otimes p_{45}(\tilde{r}_k) = \{2g-(6k-2)\}\{2g-(6k-4)\}\sum_{C\in\mathcal{D}^\ell(6k)^{(2)}} a_C \tag{22}$$

where

$$\mathcal{D}^\ell(6k)^{(2)} = \{C \in \mathcal{D}^\ell(6k); \{1,2\},\{4,5\} \in C\}.$$

The number of elements of the set $\mathcal{D}^\ell(6k)^{(2)}$ is equal to $(6k-5)!!$. Since

$$3^2\binom{2k}{2}\{2g-(6k-2)\}\{2g-(6k-4)\} = 2^2 3^2\binom{2k}{2}(g-g_0)(g-g_0-1)$$

this proves the claim for $r_k^{(2)}$.

It is now clear that similar arguments prove the remaining general cases. □

Unfortunately it is very difficult to determine the coefficients $n_j^{(p)}$ which appear in Proposition 6.1 (except for the most degenerate case $p=2k$ where there is only one non-trivial term corresponding to the disjoint union of k copies of Γ_{Z_1} which is the trivalent graph with 2 vertices and two loops). However, we can determine the coefficients of $r_k^{(p)}$ with respect to the connected types of trivalent graphs in it.

Here we consider the case of $r_k^{(1)}$. In view of (21), it is natural to make the following definition.

Definition 6.2. For each natural number k, we define c_k to be the number of linear chord diagrams C with $6k$ vertices satisfying the following two conditions: (i) $\{1,2\}\in C$ and (ii) the associated trivalent graph Γ_C is *connected*.

Lemma 6.3. *We have the equality*

$$(6k-3)!! = c_k + \sum_{i=1}^{k-1}\binom{2k-1}{2i} c_{k-i}a_i$$

so that the number c_k can be determined recursively. The first few values of c_k are

$$c_1 = 3,\ c_2 = 810,\ c_3 = 1749600,\ c_4 = 12179840400,\ c_5 = 193706418124800.$$

Proof. There are $(6k-3)!!$ linear chord diagrams C with $6k$ vertices satisfying the condition (i) above. Hence we can write

$$(6k-3)!! = c_k + \text{decomposable terms}.$$

To analize the decomposable terms, we consider the connected component of the graph Γ_C which contains the image of the chord $\{1,2\}$. We denote this component by Γ_C^0. We can classify the decomposable terms by the number v_C of vertices of Γ_C^0. The possible values for v_C are $2k-2, 2k-4, \cdots, 2$. Now we claim that the number of C with $v_C = 2k-2$ is equal to $\binom{2k-1}{2} c_{k-1} a_1$ (recall that we write a_i for the number $(6i-1)!!$). This is because of the following reason. We need $2k-3$ blocks other than the first one $\{1,2,3\}$ to make Γ_C^0 and there are $\binom{2k-1}{2k-3} = \binom{2k-1}{2}$ such choices. In each choice, there are c_{k-1} linear chord diagrams spanning $6k-6$ vertices in the $2k-2$ blocks to make Γ_C^0, just by definition of the numbers c_k. We are left with remaining two blocks and there are $a_1 = 15$ linear chord diagrams spanning the corresponding 6 vertices. This proves the claim. Similarly the number of C with $v_C = 2k-4$ is equal to $\binom{2k-1}{4} c_{k-2} a_2$ and so on. Finally the number of C with $v_C = 2$ is equal to $\binom{2k-1}{2k-2} c_1 a_{k-1}$ This completes the proof. □

Definition 6.4. Let $\gamma_k \in \mathbb{Q}[e_1, e_2, \cdots, e_k]$ denote the homogeneous polynomial of degree $2k$ which is obtained by substituting $c_i e_i$ and α_i in c_i and a_i, respectively, in the right hand side of the equality in Lemma 6.3. Namely we set

$$\gamma_k = c_k e_k + \sum_{i=1}^{k-1} \binom{2k-1}{2i} c_{k-i} e_{k-i} \alpha_i.$$

Theorem 6.5. *For any $k \geq 2$, we have the relation*

$$\alpha_k + \frac{2k}{k-1} \gamma_k = 0$$

which holds in $H^{2k}(\mathcal{M}_{3k-2}; \mathbb{Q})$.

Proof. By Proposition 5.3, we know that $\bar{\alpha}_k = 0$. Hence we have only to evaluate $\bar{\alpha}_k$ explicitly. If $g = 3k-2 = g_0 - 1$, Proposition 6.1 implies that

$$\bar{r}_k = r_k - r_k^{(1)}.$$

Since $\bar{\alpha}_k$ does not contain the Euler class e again by Proposition 5.3, we have only to identify the leading terms of the characteristic classes corresponding to r_k and $r_k^{(1)}$. By the results of §5, the former is given by the polynomial α_k. On the other hand, Proposition 6.1 and the definition of γ_k above implies that the latter is expressed as

$$-\frac{1}{2g-2} 2 \cdot 3 \binom{2k}{1} \gamma_k = -\frac{2k}{k-1} \gamma_k$$

(notice that $g - g_0 = -1$). Hence

$$\bar{\alpha}_k = \alpha_k + \frac{2k}{k-1} \gamma_k.$$

Strictly speaking, we have to put the sign $(-1)^k$ in the right hand side of the final formula above. However we omit it by the same reason as in the proof of Theorem 5.7. This completes the proof. □

Example 6.6. If $k = 2$, then by substituting $\alpha_1 = 15\, e_1$, $\alpha_2 = 9720\, e_2 + 675\, e_1^2$ and the values c_1, c_2 given in lemma 6.3 in the above formula, we obtain the relation $405(32\, e_2 + 3\, e_1^2) = 0$. Similarly we obtain the second relations for small values of k as follows. Here we normalize them so that the coefficients are all integers such that the g.c.d. is 1.

$k=2,\quad g=4,\quad 32\,e_2+3\,e_1^2=0$

$k=3,\quad g=7,\quad 4626\,e_3+369\,e_1e_2+10\,e_1^3=0$

$k=4,\quad g=10,\quad 23379840\,e_4+1185696\,e_1e_3+285120\,e_2^2+44640\,e_1^2e_2+575\,e_1^4=0.$

7. The third and the fourth relations

In this section, we describe the relation $\bar{\alpha}_k(g)=0$ given in Proposition 5.3 explicitly for the cases $g=3k-3$ and $g=3k-4$.

First we treat the case $g=3k-2=g_0-2$. By Proposition 6.1, we have

$$\bar{r}_k=r_k-r_k^{(1)}+r_k^{(2)}.$$

As in §5, to evaluate the leading terms of $\alpha_k^{(2)}(g)$, we make the following definition.

Definition 7.1. For each natural number k, we define d_k to be the number of elements $C\in\mathcal{D}^\ell(6k)$ which satisfy the following two conditions: (i) $\{1,2\},\{4,5\}\in C$ and (ii) the associated trivalent graph Γ_C is *connected*.

Lemma 7.2. *We have the equality*

$$\begin{aligned}(6k-5)!! =& d_k+\sum_{i=1}^{k-1}\binom{2k-2}{2i}d_{k-i}a_i\\ &+\sum_{i=1}^{k-1}\binom{2k-2}{2i-1}\sum_{j=1}^{k-i}\binom{2k-2i-1}{2j-1}c_ic_ja_{k-i-j}\end{aligned}$$

so that the number d_k can be determined recursively (we set $a_0=1$). The first few values of d_k are

$$d_1=1,\ d_2=72,\ d_3=97200,\ d_4=503884800,\ d_5=6430955731200.$$

Proof. There are $(6k-5)!!$ elements $C\in\mathcal{D}^\ell(6k)$ which satisfy the condition (i) above. Hence we can write

$$(6k-5)!!=d_k+\text{decomposable terms}.$$

The decomposable terms are classified into two types. One is the type where the images of two chords $\{1,2\},\{4,5\}$ in Γ_C belong to the same connected component and the other is the type where they belong to different components, say Γ_C^1 and Γ_C^2, of Γ_C. A similar argument as in the proof of Lemma 6.3 implies that the number of the former types is given by

$$\sum_{i=1}^{k-1}\binom{2k-2}{2i}d_{k-i}a_i.$$

Also it is easy to see that the number of the latter types is given by

$$\sum_{i=1}^{k-1}\binom{2k-2}{2i-1}\sum_{j=1}^{k-i}\binom{2k-2i-1}{2j-1}c_ic_ja_{k-i-j}.$$

Here the term $c_i c_j a_{k-i-j}$ corresponds to the case where Γ_C^1 has $2i$ vertices and Γ_C^2 has $2j$ vertices. The two binomial coefficients correspond to the number of choices of $2i-1$ blocks out of $2k-2$ blocks for Γ_C^1 and the number of choices of $2j-1$ blocks out of the remaining $2k-2i-1$ blocks for Γ_C^2. This completes the proof. □

Definition 7.3. Let $\delta_k \in \mathbb{Q}[e_1, e_2, \cdots, e_k]$ denote the homogeneous polynomial of degree $2k$ which is obtained by substituting $c_i e_i$, $d_i e_i$ and α_i in c_i, d_i and a_i, respectively, in the right hand side of the equality in Lemma 7.2. Namely we set

$$\begin{aligned}\delta_k = d_k e_k &+ \sum_{i=1}^{k-1} \binom{2k-2}{2i} d_{k-i} e_{k-i} \alpha_i \\ &+ \sum_{i=1}^{k-1} \binom{2k-2}{2i-1} \sum_{j=1}^{k-i} \binom{2k-2i-1}{2j-1} c_i c_j e_i e_j \alpha_{k-i-j}.\end{aligned}$$

Theorem 7.4. *For any $k \geq 2$, we have the relation*

$$\alpha_k + \frac{12k}{3k-4}\gamma_k + \frac{18k(2k-1)}{(3k-4)^2}\delta_k = 0$$

which holds in $H^{2k}(\mathcal{M}_{3k-3};\mathbb{Q})$.

Proof. Proof is similar to that of Theorem 6.5. Here we only check the coefficients in the formula. In this case we have $g = 3k-3$ so that

$$\frac{1}{2g-2} 2 \cdot 3 \binom{2k}{1} (g - g_0) = -\frac{12k}{3k-4}$$

and

$$\frac{1}{(2g-2)^2} 2^2 3^2 \binom{2k}{2} (g-g_0)(g-g_0+1) = \frac{18k(2k-1)}{(3k-4)^2}.$$

This completes the proof. □

Example 7.5. Here are explicit third relations for $k = 3, 4$ where we normalize them so that the coefficients are all integers such that the g.c.d. is 1.

$$\begin{aligned} &k=3, \quad g=6, \quad 566280\, e_3 + 55512\, e_1 e_2 + 1831\, e_1^3 = 0 \\ &k=4, \quad g=9, \quad 3007814342400\, e_4 + 174574650600\, e_1 e_3 + 41168962800\, e_2^2 \\ &\qquad\qquad\qquad + 7421803200\, e_1^2 e_2 + 108226125\, e_1^4 = 0. \end{aligned}$$

We compare the above results with those of Faber [5]. If $g = 6$, Faber showed that

$$e_1 e_2 = -\frac{127}{2304} e_1^3, \quad e_3 = \frac{5}{2304} e_1^3.$$

It is easy to see that the first equality above is consistent with this. Next we consider the second equality for $g = 9$ above, whose left hand side we denote by (A). Faber describes the tautological algebra $\mathcal{R}^*(\mathbf{M}_9)$ of the genus 9 moduli space explicitly. Among his relations in degree 8, let us consider the following two:

$$\begin{aligned} &(B)\ 3644694\, e_1 e_3 + 749412\, e_2^2 + 265788\, e_1^2 e_2 + 5195\, e_1^4 = 0 \\ &(C)\ 1399562496\, e_4 + 2453760\, e_2^2 - 2470320\, e_1^2 e_2 - 65425\, e_1^4 = 0 \end{aligned}$$

where (B) and (C) denote the left hand sides of the corresponding equalities. Then we have

$$202483\,(A) - 9698591700\,(B) - 435158325\,(C) = 0$$

which shows that this case is also consistent with that of Faber.

Next we consider the case $g = 3k-4 = g_0 - 3$. To evaluate the leading terms of $\alpha_k^{(3)}(g)$, we make the following definition.

Definition 7.6. For each integer $k \geq 2$, we define f_k to be the number of elements $C \in \mathcal{D}^\ell(6k)$ which satisfy the following two conditions: (i) $\{1,2\},\{4,5\},\{7,8\} \in C$ and (ii) the associated trivalent graph Γ_C is *connected.*

Lemma 7.7. *We have the equality*

$$\begin{aligned}
&(6k-7)!!\\
&= f_k + \sum_{i=1}^{k-2}\binom{2k-3}{2i} f_{k-i}a_i + 3\Big\{c_{k-1} + \sum_{i=1}^{k-2}\binom{2k-3}{2i} c_{k-i-1}a_i\Big\}\\
&\quad + 3\sum_{i=2}^{k-1}\binom{2k-3}{2i-2}\sum_{j=1}^{k-i}\binom{2k-2i-1}{2j-1} d_i c_j a_{k-i-j}\\
&\quad + \sum_{i=1}^{k-2}\binom{2k-3}{2i-1}\sum_{j=1}^{k-i-1}\binom{2k-2i-2}{2j-1}\sum_{\ell=1}^{k-i-j}\binom{2k-2i-2j-1}{2\ell-1} c_i c_j c_\ell a_{k-i-j-\ell}
\end{aligned}$$

so that the number f_k can be determined recursively. The first few values of f_k are

$$f_2 = 6,\ f_3 = 5184,\ f_4 = 20412000,\ f_5 = 211631616000.$$

Proof. There are $(6k-7)!!$ elements $C \in \mathcal{D}^\ell(6k)$ which satisfy the condition (i) above. Hence we can write

$$(6k-7)!! = f_k + \text{decomposable terms}.$$

This time, the decomposable terms are classified into four types.

The first one is the type where the images of the 3 chords $\{1,2\},\{4,5\},\{7,8\}$ in Γ_C belong to the same connected component. The number of such types is given by

$$\sum_{i=1}^{k-2}\binom{2k-3}{2i} f_{k-i}a_i.$$

The second one is the type where there is a chord connecting two vertices in $\{3,6,9\}$. There are 3 such choices so that the number of this types is given by

$$3\Big\{c_{k-1} + \sum_{i=1}^{k-2}\binom{2k-3}{2i} c_{k-i-1}a_i\Big\}$$

by the definition of the numbers c_i (see Definition 6.2). The third one is the type where there are no chords connecting the 3 vertices $\{3,6,9\}$ and the images of the 3 chords $\{1,2\},\{4,5\},\{7,8\}$ in Γ_C belong to two connected components of Γ_C. The number of this types is given by

$$3\sum_{i=2}^{k-1}\binom{2k-3}{2i-2}\sum_{j=1}^{k-i}\binom{2k-2i-1}{2j-1} d_i c_j a_{k-i-j}$$

where 3 is the number of choices of two chords out of the above mentioned 3 chords.

Finally, the fourth one is the type where the images of $\{1,2\},\{4,5\},\{7,8\}$ in Γ_C belong to mutually different connected components. The number of this types is given by

$$\sum_{i=1}^{k-2}\binom{2k-3}{2i-1}\sum_{j=1}^{k-i-1}\binom{2k-2i-2}{2j-1}\sum_{\ell=1}^{k-i-j}\binom{2k-2i-2j-1}{2\ell-1}c_ic_jc_\ell a_{k-i-j-\ell}.$$

This completes the proof. $\square$

Definition 7.8. Let $\theta_k\in\mathbb{Q}[e_1,e_2,\cdots,e_k]$ denote the homogeneous polynomial of degree $2k$ which is obtained by substituting c_ie_i, d_ie_i, f_ie_i and α_i in c_i, d_i, f_i and a_i, respectively, in the right hand side of the equality in Lemma 7.7 and further we multiply the third term by an extra e_1. Namely we set

$$\begin{aligned}
&\theta_k = f_ke_k\\
&+\sum_{i=1}^{k-2}\binom{2k-3}{2i}f_{k-i}e_{k-i}\alpha_i+3e_1\Big\{c_{k-1}e_{k-1}+\sum_{i=1}^{k-2}\binom{2k-3}{2i}c_{k-i-1}e_{k-i-1}\alpha_i\Big\}\\
&+3\sum_{i=2}^{k-1}\binom{2k-3}{2i-2}\sum_{j=1}^{k-i}\binom{2k-2i-1}{2j-1}d_ie_ic_je_j\alpha_{k-i-j}\\
&+\sum_{i=1}^{k-2}\binom{2k-3}{2i-1}\sum_{j=1}^{k-i-1}\binom{2k-2i-2}{2j-1}\sum_{\ell=1}^{k-i-j}\binom{2k-2i-2j-1}{2\ell-1}c_ic_jc_\ell e_ie_je_\ell\alpha_{k-i-j-\ell}.
\end{aligned}$$

Theorem 7.9. *For any $k\geq 2$, we have the relation*

$$\alpha_k+\frac{18k}{3k-5}\gamma_k+\frac{54k(2k-1)}{(3k-5)^2}\delta_k+\frac{54k(2k-1)(2k-2)}{(3k-5)^3}\theta_k=0$$

which holds in $H^{2k}(\mathcal{M}_{3k-4};\mathbb{Q})$.

Proof. Proof is given in the same way as that of Theorem 7.4. We only mention that, in the definition of θ_k above, we put e_1 in the term corresponding to the second type of linear chord diagrams. This is because there arises the trivalent graph Γ_{Z_1} (which has two vertices and two loops) whose leading term is $-e_1$. $\square$

Example 7.10. Here are explicit fourth relations for $k=3,4$ where we normalize them so that the coefficients are all integers such that the g.c.d. is 1.

$$\begin{aligned}
&k=3,\quad g=5,\quad 602712\,e_3+77382\,e_1e_2+3281\,e_1^3=0\\
&k=4,\quad g=8,\quad 3551361408\,e_4+242374752\,e_1e_3+55847232\,e_2^2\\
&\qquad\qquad\qquad +11900736\,e_1^2e_2+200699\,e_1^4=0.
\end{aligned}$$

If we substitute the relation $e_2=-\frac{5}{72}e_1^2$, which is the first relation for $k=2$ $(g=5)$ (see Example 5.9), in the first equation above, then we obtain the known relation

$$288\,e_3=e_1^3$$

(cf. [5]).

8. The i-th relations

In this section, we generalize the considerations in previous sections. We would like to describe the relation $\bar{\alpha}_k(g) = 0$ given in Proposition 5.3 explicitly for the general cases $g = 3k - 1 - i$ $(i = 1, 2, \cdots)$. To analize the leading terms of $\alpha_k^{(i)}(g)$, we make the following definition.

Definition 8.1. For each positive integer k and each i with $1 \leq i \leq 2k$, we define $c_k^{(i)}$ to be the number of elements $C \in \mathcal{D}^\ell(6k)$ which satisfy the following two conditions: (i) $\{1,2\}, \{4,5\}, \cdots, \{3i-2, 3i-1\} \in C$ and (ii) the associated trivalent graph Γ_C is *connected*. We set $c_k^{(i)} = 0$ for $i > 2k$.

Remark 8.2. We denoted the numbers $c_k^{(1)}, c_k^{(2)}, c_k^{(3)}$ by c_k, d_k, f_k respectively in §6 and §7. Also if $i \geq 2$, it is easy to see that $c_k^{(i)} > 0$ if and only if $i \leq k+1$. Thus if we fix i (≥ 2), then we have a sequence of natural numbers

$$c_{i-1}^{(i)}, c_i^{(i)}, c_{i+1}^{(i)}, \cdots .$$

For $i = 0$, it may be natural to set $c_k^{(0)} = b_k$ (see Definition 5.5). Hence the above sequences of numbers are generalizations of the sequence $b_1, b_2, \cdots$. It might be worthwhile to investigate number theoretical properties of these sequences of natural numbers because the prime decompositions of the first several terms of them seem to be interesting.

Lemma 8.3. *We have the equality*

$$\begin{aligned}
&(6k-2i-1)!! \\
=&c_k^{(i)} + \sum_{s=1}^{k-i+1} \binom{2k-i}{2s} c_{k-s}^{(i)} a_s \\
&+ \sum_{i_0=0}^{[\frac{i}{2}]} \binom{i}{2i_0} (2i_0-1)!! \times \sum_{\substack{j_1 i_1 + \cdots + j_s i_s = i - 2i_0 \\ i_1 < \cdots < i_s}} \frac{(i-2i_0)!}{(i_1!)^{j_1} \cdots (i_s!)^{j_s} j_1! \cdots j_s!} \\
&\times \sum_{\substack{\ell_u^t \\ t=1,\cdots,s, u=1,\cdots,j_t}} \frac{(2k-i)!}{(2\ell_1^1 - i_1)! \cdots (2\ell_{j_1}^1 - i_1)! \cdots (2\ell_1^s - i_s)! \cdots (2\ell_{j_s}^s - i_s)!} \\
&\times \frac{1}{\{2(k - i_0 - \ell_1^1 - \cdots - \ell_{j_1}^1 - \cdots - \ell_1^s - \cdots - \ell_{j_s}^s)\}!} \\
&\times \prod_{\substack{t=1,\cdots,s \\ u=1,\cdots,j_t}} c_{\ell_u^t}^{(i_t)} \times a_{k-i_0-\ell_1^1-\cdots-\ell_{j_1}^1-\cdots-\ell_1^s-\cdots-\ell_{j_s}^s}
\end{aligned}$$

so that the number $c_k^{(i)}$ can be determined recursively (in the case $i = 1$ we set $c_0^{(1)} = 0$). Here the summations are taken as follows. For the second summation, if $i_0 = 0$ we understand the coefficient

$$\binom{i}{2i_0}(2i_0-1)!!$$

to be 1 and the summation is taken over all partitions $i_1 j_1 + \cdots + i_s j_s = i$ of i with $j_1 + \cdots + j_s > 1$. However for the case $i_0 > 0$, we also take the trivial partition *of i, namely the case $s = 1, j_1 = 1$, into account. Also if $i = 2i_0$, then there are no partitions of $i - 2i_0 = 0$ but we consider the single term a_{k-i_0} (except for the exceptional case $i = 2$ where we ignore this term because it is already counted in the first summation in the above formula as $c_1^{(2)} a_{k-1}$). Finally the last summation is taken over all ℓ_u^t with $k - i_0 - \ell_1^1 - \cdots - \ell_{j_1}^1 - \cdots - \ell_1^s - \cdots - \ell_{j_s}^s \geq 0$.*

Proof. There are $(6k - 2i - 1)!!$ elements $C \in \mathcal{D}^\ell(6k)$ which satisfy the condition (i) above. Hence we can write

$$(6k - 2i - 1)!! = c_k^{(i)} + \text{decomposable terms}.$$

We have to enumerate the decomposable terms. We first consider those C where the images of the i chords $\{1, 2\}, \{4, 5\}, \cdots, \{3i - 2, 3i - 1\}$ in Γ_C belong to the same connected component. The number of such linear chord diagrams is given by

$$\sum_{s=1}^{k-i+1} \binom{2k - i}{2s} c_{k-s}^{(i)} a_s.$$

Next we classify the remaining types of C by the way how the images of the above i chords in Γ_C are distributed to various connected components of it. Let i_0 denote the number of chords which connect $2i_0$ vertices out of $\{3, 6, \cdots, 3i\}$ in pairs. Thus if we fix $i_0 > 0$, there are $\binom{i}{2i_0}(2i_0 - 1)!!$ such choices. They give rise to i_0 copies of Γ_{Z_1} within Γ_C. Then still $i - 2i_0$ vertices among $\{1, \cdots, 3i\}$ are left over and we have to enumerate how these vertices together with the remaining vertices $\{3i + 1, \cdots, 6k\}$ are joined into pairs by chords. These are described by certain partition

$$i - 2i_0 = i_1 j_1 + \cdots + i_s j_s \quad (i_1 < \cdots < i_s,\ j_\ell > 0) \tag{23}$$

of the number $i - 2i_0$ and additional system of numbers

$$\ell_u^t \quad (t = 1, \cdots, s, u = 1, \cdots, j_t)$$

such that $2\ell_u^t > i_t$ for any t, u and

$$2k - 2i_0 - 2\ell_1^1 - \cdots - 2\ell_{j_1}^1 - \cdots - 2\ell_1^s - \cdots - 2\ell_{j_s}^s \geq 0.$$

Any such partition together with the above system of numbers corresponds to a linear chord diagram C with the following property. Namely the remaining $2k - 2i_0$ blocks are decomposed into the disjoint union of $2\ell_1^1, \cdots, 2\ell_{j_1}^1, \cdots, 2\ell_1^s, \cdots, 2\ell_{j_s}^s$ blocks and the left over in such a way that for any $t = 1, \cdots, s$ and $u = 1, \cdots, j_t$, the corresponding $2\ell_u^t$ blocks contain exactly i_t vertices of the former type and the chords are spanned within them such that the associated trivalent graph is connected. If $i_0 = 0$, we have to assume that the partition (23) is non-trivial, namely $j_1 + \cdots + j_s > 1$ because the trivial partition $i = i$ is already counted. Finally it is easy to see that the number of such C can be expressed in terms of various binomial coefficients as in our formula. More precisely, the term

$$\frac{(i - 2i_0)!}{(i_1!)^{j_1} \cdots (i_s!)^{j_s} j_1! \cdots j_s!}$$

represents the number of decompositions of the remaining $i-2i_0$ vertices among $\{3,6,\cdots,3i\}$ into the prescribed partition $i_1j_1+\cdots+i_sj_s=i-2i_0$ and the term

$$\frac{(2k-i)!}{(2\ell_1^1-i_1)!\cdots(2\ell_{j_1}^1-i_1)!\cdots(2\ell_1^s-i_s)!\cdots(2\ell_{j_s}^s-i_s)!}$$
$$\times\frac{1}{\{2(k-i_0-\ell_1^1-\cdots-\ell_{j_1}^1-\cdots-\ell_1^s-\cdots-\ell_{j_s}^s)\}!}$$

counts the number of decompositions of the remaining vertices according to the prescribed additional pattern. This completes the proof. □

Definition 8.4. We define a homogeneous polynomial

$$\gamma_k^{(i)}\in\mathbb{Q}[e_1,e_2,\cdots,e_k]$$

of degree $2k$ by replacing $c_\ell^{(j)}$, a_j, in the right hand side of the equality in Lemma 8.3, by $c_\ell^{(j)}e_\ell$ and α_j, respectively and also we multiply each term corresponding to i_0 by $e_1^{i_0}$.

Theorem 8.5. *Let k be any positive integer. Then for any i with $0\le i\le 3k-3$ (so that $2\le 3k-i-1\le 3k-1$), we have the relation*

$$\alpha_k+\frac{3i}{3k-i-2}\binom{2k}{1}\gamma_k^{(1)}+\frac{3^2i(i-1)}{(3k-i-2)^2}\binom{2k}{2}\gamma_k^{(2)}+\cdots+\frac{3^ii!}{(3k-i-2)^i}\binom{2k}{i}\gamma_k^{(i)}=0$$

which holds in $H^{2k}(\mathcal{M}_{3k-i-1};\mathbb{Q})$. Here note that $\gamma_k^{(i)}=0$ for $i>k+1$.

Proof. Similar arguments as in the special cases considered in previous sections (§5, 6, 7) show that the above formula gives the explicit expression of the characteristic class $\bar{\alpha}_k(g)$. Here observe that the alternating negative signs in (17) are exactly cancelled by those in the formula of Proposition 6.1 for any $g<g_0$ so that all the coefficients in our formula above are positive. □

Corollary 8.6. *Let k be any positive integer. Then for any genus g with $2\le g\le 3k-1$, the k-th Mumford-Morita-Miller class $e_k\in H^{2k}(\mathcal{M}_g;\mathbb{Q})$ can be expressed as an explicit polynomial in lower classes $e_1,\cdots,e_{k-1}$.*

Proof. It is clear from the definitions of $c_k^{(i)}$ and $\gamma_k^{(i)}$ that any non-trivial coefficient of them is a positive integer. Hence the coefficient of e_k which appears in the relation given in Theorem 8.5 is also positive for all i. The claim follows from this immediately. □

Example 8.7. Explicit computation shows that $c_3^{(4)}=216$ and our relation for the case $k=3,i=4$ $(g=4)$ is given by

$$232776\,e_3+43096\,e_1e_2+2553\,e_1^3=0.$$

If we substitute the second relation $e_2=-\frac{3}{32}e_1^2$ (see Example 6.6) in the above equality, we obtain

$$e_3=\frac{661}{103456}e_1^3=\frac{661}{2^5\cdot 53\cdot 61}e_1^3.$$

On the other hand, Mumford's algorithm in [24] to express κ_i $(i \geq g-1)$ in terms of $\kappa_1, \cdots, \kappa_{g-2}$, implies

$$\kappa_3 = \frac{85}{8784}\kappa_1^3 = \frac{5 \cdot 17}{2^4 \cdot 3^2 \cdot 61}\kappa_1^3.$$

Hence the above two relations are independent and we can conclude the vanishing of both e_3 and e_1^3. This is consistent with the fact that $\mathcal{R}^6(\mathcal{M}_4) = 0$.

9. Concluding remarks

In this paper, we have used only the relation $\bar{\alpha}_k(g) = 0$ for $g \leq 3k-1$. However Proposition 5.3 shows that a single unstable relation $\tilde{r}_k = 0$ in $(H_{\mathbb{Q}}^{\otimes 6k})^{Sp}$ yields more relations. Here we give a few examples.

Example 9.1. We consider the case $k = 1$. Then there are 15 elements in $\mathcal{D}^\ell(6)$. Let Γ_{Z_1} and Γ_{Z_2} be the trivalent graphs with 2 vertices such that the former has two loops while the latter is the *theta* graph. Then it is easy to see that among the 15 trivalent graphs Γ_C $(C \in \mathcal{D}^\ell(6))$, there are 9 copies of Γ_{Z_1} and 6 copies of Γ_{Z_2}. On the other hand, we proved that

$$\alpha_{\Gamma_{Z_1}} = -e_1 - 4g(g-1)e, \quad \alpha_{\Gamma_{Z_2}} = -e_1 + 6ge$$

(see [20]). Hence

$$\alpha_1(g) = -15e_1 - 36g(g-2)e$$

and the first relation for $g = 2$ is $e_1 = 0$.

Example 9.2. If $k = 2$, a long computation using Proposition 3.6 and the results in [12] shows that

$$\begin{aligned}
\alpha_2(g) =& 27\{5(72\, e_2 + 5\, e_1^2) + 120(g-5)(g-3)ee_1 \\
&\quad + 48(g-5)(g-3)(3g^2 - 8g - 1)e^2\} \\
\alpha_2^{(1)}(g) =& \frac{648}{2g-2}(g-5)\{5(6\, e_2 + e_1^2) + 2(16g^2 - 76g + 63)ee_1 \\
&\quad + 12(g-1)(4g^3 - 24g^2 + 32g + 5)e^2\} \\
\alpha_2^{(2)}(g) =& \frac{648}{(2g-2)^2}(g-5)(g-4)\{24\, e_2 + 11\, e_1^2 + 4(20g^2 - 53g + 21)ee_1 \\
&\quad + 48(g-1)(3g^3 - 10g^2 + 5g + 1)e^2\} \\
\alpha_2^{(3)}(g) =& \frac{2592}{(2g-2)^3}(g-5)(g-4)(g-3)\{2\, e_2 + 3\, e_1^2 + 6(4g^2 - 6g + 1)ee_1 \\
&\quad + 4(g-1)(12g^3 - 20g^2 + 4g + 1)e^2\} \\
\alpha_2^{(4)}(g) =& \frac{3888}{(2g-2)^4}(g-5)(g-4)(g-3)(g-2)\{e_1^2 + 8g(g-1)ee_1 \\
&\quad + 16g^2(g-1)^2e^2\}.
\end{aligned}$$

By Proposition 5.3, we know that all the above elements vanish for any $g \leq 5$. The case $g = 5$ gives the first relation $72\, e_2 + 5\, e_1^2$ $(g = 5)$ given in §5. If $g = 4$, $\alpha_2(4) - \alpha_2^{(1)}(4)$ gives the second relation $32\, e_2 + 3\, e_1^2 = 0$ given in §6. One more relation is

$$7\, e_2 - 9\, ee_1 - 54\, e^2 = 0 \ \in \mathcal{R}^4(\mathcal{M}_{4,*}).$$

It is amusing to observe here the following fact. Let $\pi_* : \mathcal{R}^*(\mathcal{M}_{4,*})\to\mathcal{R}^{*-2}(\mathcal{M}_4)$ be the integration along the fiber. If we apply π_* to the above element, we obtain the trivial relation $54\,e_1 - 54\,e_1 = 0$ (notice that $\pi_*(e) = -6$) while if we apply it to e times the above element, then we obtain the second relation again. Similar computations for the cases $g = 3$ and $g = 2$ show that the above relations give a complete set of relations for $\mathcal{R}^*(\mathcal{M}_{g,*})$ as well as $\mathcal{R}^*(\mathcal{M}_g)$ for these genera.

Remark 9.3. It seems very likely that we can obtain many more (and possibly all) relations in the tautological algebras $\mathcal{R}^*(\mathcal{M}_{g,*})$ and $\mathcal{R}^*(\mathcal{M}_g)$ in cohomology of the moduli space of curves by making use of deeper unstable relations in $(\Lambda^*(\Lambda^3 H_{\mathbb{Q}}))^{Sp}$ as well as in $(\Lambda^* U_{\mathbb{Q}})^{Sp}$. Here a result of Mihailovs in [16], which gives a basis of $(H_{\mathbb{Q}}^{\otimes 2k})^{Sp}$ for *any* genus g, would be useful.

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References

1. D. Bar-Natan, *On the Vassiliev knot invariants*, Topology 34 (1995), 423–472.
2. C. Faber, *Chow rings of moduli spaces of curves I: The Chow ring of* $\overline{\mathcal{M}}_3$, Ann. of Math. 132 (1990), 331–419.
3. C. Faber, *Chow rings of moduli spaces of curves II: Some results on the Chow ring of* $\overline{\mathcal{M}}_4$, Ann. of Math. 132 (1990), 421–449.
4. C. Faber, *A non-vanishing result for the tautological ring of* $\mathcal{M}_g$, preprint 1995.
5. C. Faber, *A conjectural description of the tautological ring of the moduli space of curves*, in *Moduli of Curves and Abelian Varieties*, Carel Faber and Eduard Looijenga, editors, Vieweg 1999.
6. R. Hain, E. Looijenga, *Mapping class groups and moduli space of curves*, Proc. Symp. Pure Math. 62.2 (1997), 97–142.
7. J. Harer, *Stability of the homology of the mapping class group of an orientable surface*, Ann. of Math. 121 (1985), 215–249.
8. J. Harer, *Improved stability for the homology of the mapping class groups of surfaces*, Duke University preprint 1993.
9. J. Harris, I. Morrison, *Moduli of Curves*, Graduate Texts in Math. 187, Springer Verlag 1998.
10. N. Kawazumi, *A generalization of the Morita-Mumford classes to extended mapping class groups for surfaces*, Invent. Math. 131 (1998), 137–149.
11. N. Kawazumi, S. Morita, *The primary approximation to the cohomology of the moduli space of curves and cocycles for the stable characteristic classes*, Math. Res. Letters 3 (1996), 629–641.
12. N. Kawazumi, S. Morita, *The primary approximation to the cohomology of the moduli space of curves and cocycles for the Mumford-Morita-Miller classes*, preprint.
13. M. Kontsevich, *Formal (non-)commutative symplectic geometry*, in *The Gelfand Mathematical Seminars 1990-1992*, Birkhäuser Verlag 1993, 173–188.
14. M. Kontsevich, *Feynman diagrams and low-dimensional topology*, in *Proceedings of the First European Congress of Mathematicians*, Vol II, Paris 1992, Progress in Math. 120, Birkhäuser Verlag 1994, 97–121.
15. E. Looijenga, *On the tautological ring of* $\mathcal{M}_g$, Invent. Math. 121 (1995), 411-419.
16. A. Mihailovs, *Symplectic tensor invariants, wave graphs and S-tris*, preprint 1998.
17. E.Y. Miller, *The homology of the mapping class group*, J. Differential Geometry 24 (1986), 1–14.
18. S. Morita, *Characteristic classes of surface bundles*, Bull. Amer. Math. Soc. 11 (1984), 386–388.
19. S. Morita, *Characteristic classes of surface bundles*, Invent. Math. 90 (1987), 551-577.
20. S. Morita, *The structure of the mapping class group and characteristic classes of surface bundles*, in *Mapping Class Groups and Moduli Spaces of Riemann Surfaces*, C.-F. Bödigheimer and R. Hain, editors, Contemporary Math. 150 (1993), 303–315.

21. S. Morita, *The extension of Johnson's homomorphism from the Torelli group to the mapping class group*, Invent. Math. 111 (1993), 197–224.
22. S. Morita, *A linear representation of the mapping class group of orientable surfaces and characteristic classes of surface bundles*, in *Proceedings of the Taniguchi Symposium on Topology and Teichmüllller Spaces* held in Finland, July 1995, World Scientific 1996, 159–186.
23. S. Morita, *Structure of the mapping class groups of surfaces: a survey and a prospect*, Geometry and Topology Monographs 2 (1999), *Proceedings of the Kirbyfest*, 349–406.
24. D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in *Arithmetic and Geometry*, Progress in Math. 36 (1983), 271–328.

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