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Conductor is a numerical invariant of a variety over a local field measuring the wild ramification of the inertia action on the ℓ-adic etale cohomology. In [3], S.Bloch proposes a conjectural formula, Conjecture 1.9, which we call the conductor formula of Bloch. To formulate it, he defines another numerical invariant as the degree of the self-intersection class, which is defined using the localized chern class of the sheaf of differential 1-forms of a regular proper model of the variety. The conductor formula asserts that the conductor of a regular proper model of a variety over a local field is equal to the minus of the degree of the self-intersection class. It enables us to compute a Galois theoretic invariant, the conductor, of ramification in terms of a de Rham theoretic invariant, the self-intersection class, of degeneration. In the same paper, he proves the formula for curves over a local field. For a finite extension of a local field, the conductor formula is nothing but the classical conductor-discriminant formula in algebraic number theory. For an elliptic curve, the formula is known in [29] Corollary 2 of Theorem 1 to be equivalent to the Tate-Ogg formula [26] for the relation between the conductor and the discriminant.

In this paper, we prove the conductor formula in arbitrary dimension under a rather mild hypothesis. To describe the main results, we introduce some notations. Detailed explanation is given in the text. Let K be a discrete valuation field with perfect residue field F and let X_K be a proper smooth scheme over K of dimension d. The Swan conductor $\mathrm{Sw}(X_K/K)$ of X_K is defined to be the alternating sum $\mathrm{Sw}(X_K/K)=\sum_{q=0}^{2d}(-1)^q \mathrm{Sw} H^q(X_{\bar K},\mathbf{Q}_\ell)$ of the Swan conductor of the ℓ-adic etale cohomology for a prime ℓ different from the characteristic p of F. The Swan conductor of an ℓ-adic representation V is defined to be the intertwining number

$$\mathrm{Sw}(V)=\frac{1}{[L:K]}\sum_{\sigma\in P_{L/K}}\mathrm{sw}_{L/K}(\sigma)\mathrm{Tr}(\sigma:V)$$

by taking a sufficiently large finite Galois extension L of K, where $\mathrm{sw}_{L/K}(\sigma)$ denotes the Swan character and $P_{L/K}$ denotes the wild inertia subgroup of $\mathrm{Gal}(L/K)$. For a proper flat and regular scheme X over $S=\mathrm{Spec}\ O_K$ such that $X\otimes_{O_K}K=X_K$, the Artin conductor $\mathrm{Art}(X/O_K)$ is defined by

$$\mathrm{Art}(X/O_K)=\chi(X_{\bar K})-\chi(X_{\bar F})+\mathrm{Sw}(X_K/K).$$

In the right hand side, χ denotes the ℓ-adic Euler number.

The localized self-intersection class $(\Delta_X,\Delta_X)_S\in CH_0(X_F)$ is defined as $(-1)^n$-times the localized chern class $c_n{}^X_{X_F}(\Omega^1_{X/O_K})\cap[X]$ of the coherent O_X-module Ω^1_{X/O_K} where $n=d+1$ is the dimension of X. Its explicit computation is given in Proposition 1.16. Let $\deg : CH_0(X_F) \to CH_0(F) = \mathbf{Z}$ be the degree map. Bloch formulates the following in [3].

Conjecture 1.9 *Let K be a discrete valuation field with perfect residue field F and let X be a proper flat and regular scheme over O_K with smooth generic fiber. Then we have*

$$\mathrm{Art}(X/O_K) = -\deg(\Delta_X, \Delta_X)_S.$$

If $\dim X_K = 1$, it is proved by him in the same paper [3]. The main result of this paper is the following.

Theorem 1.10 *Let K and X be as in Conjecture 1.9. Assume that the reduced closed fiber $(X_F)_{\mathrm{red}}$ is a divisor with normal crossings. Then Conjecture 1.9 is true.*

Under a stronger assumption that the multiplicities are prime to the residue characteristic, Theorem 1.10 is proved in [2] and [5] independently. In [5], an application of the conductor formula to the relation between an Arakelov theoretic invariant of the de Rham cohomology with the signs in the functional equations of Artin L-functions of arithmetic schemes is studied. In a geometric equi-characteristic situation, the conductor formula is studied in [17] (cf. [8] Example 14.1.5).

If we could assume an embedded resolution in a strong sense for the reduced closed fiber, Conjecture 1.9 would be a consequence of Theorem 1.10. Let X be as in Conjecture 1.9 and assume that there exists a sequence of blowing-ups $X' = X_m \to \cdots \to X_0 = X$ at regular closed subschemes supported in the closed fibers such that the reduced closed fiber $X'_{F,\mathrm{red}}$ has normal crossings. Then Theorem 1.10 applied to X' together with Proposition 1.12 implies Conjecture 1.9 for X.

By the same argument, we also prove an analoguous formula for an algebraic correspondence. Let X_K be a proper smooth scheme of dimension d over K and ℓ be a prime number different from the characteristic of the residue field F as above. For an algebraic correspondence $\Gamma \in CH_d(X_K \times_K X_K)$, let Γ^* be the endomorphism of $H^*(X_{\bar K}, \mathbf{Q}_\ell)$ induced by the cycle class of Γ. We put $\mathrm{Sw}(\Gamma, X_K/K) = \sum_{q=0}^{2d} (-1)^q \mathrm{Sw}(\Gamma^* : H^q(X_{\bar K}, \mathbf{Q}_\ell))$. For an endomorphism f of an ℓ-adic representation V, its Swan conductor is defined by

$$\mathrm{Sw}(f : V) = \frac{1}{[L:K]} \sum_{\sigma \in P_{L/K}} \mathrm{sw}_{L/K}(\sigma) \mathrm{Tr}(f \circ \sigma : V)$$

by taking a sufficiently large finite Galois extension L of K.

Let X be a proper and flat regular scheme over $S = \mathrm{Spec}\ O_K$ such that $X \otimes_{O_K} K = X_K$ and that the reduced closed fiber $X_{F,\mathrm{red}}$ has simple normal crossings. In §2.3, we define the localized intersection product $[[\ , X]] : Gr^F_\bullet G(X_K \times_K X_K) \to Gr^F_{\bullet - d} G(X_F)$ on the graded quotients of the Grothendieck groups of coherent sheaves.

Theorem 3.25 *Let K be as above and ℓ be a prime number different from the characteristic of the residue field. Let X_K be a proper smooth scheme of dimension $d = \dim X_K$ and $\Gamma \in CH^d(X_K \times_K X_K)$ be an algebraic correspondence on X_K.*

1. *The Swan conductor $\mathrm{Sw}(\Gamma, X_K/K)$ is a rational number independent of ℓ.*

2. Let X be a proper and flat regular scheme over O_K such that $X \otimes_{O_K} K = X_K$ and that the reduced closed fiber $(X_F)_{\rm red}$ is a divisor with simple normal crossings. Let $[[\Gamma, X]] \in Gr_0^F G(X_F)$ be the image by the composition map $CH_d(X_K \times_K X_K) \to Gr_d^F G(X_K \times_K X_K) \stackrel{[[\ ,X]]}{\to} Gr_0^F G(X_F)$. Then we have an equality of integers

$$\mathrm{Sw}(\Gamma, X_K/K) = -\deg[[\Gamma, X]].$$

Theorem 3.25 is a generalization to higher dimension of a logarithmic version of the formulas in [21] and [1]. The localized product in the right hand side is studied in an unpublished preprint [19] when Γ is the graph of an "admissible" automorphism (cf. Lemma 3.27).

Main ingredients of the proof of the two theorems are the following.

1. Log version of the conductor formula.
2. K-theoretic localized intersection theory.
3. Log diagonal map and log Lefschetz trace formula.

Each of them is the subject of the first three sections respectively. An outline of the proof, completed in the final section, of the conductor formula is summarized as follows.

First, we show that Theorem 1.10 is equivalent to its log version, Theorem 1.15. It claims the equality

$$\mathrm{Sw}(X_K/K) = -\deg(\Delta_X, \Delta_X)_S^{\log}.$$

The logarithmic self-intersection class $(\Delta_X, \Delta_X)_S^{\log} \in CH_0(X_F)$ is defined by replacing Ω^1_{X/O_K} in the definition of $(\Delta_X, \Delta_X)_S$ by the sheaf $\Omega^1_{X/O_K}(\log/\log)$ of differential 1-forms with log poles. Second, we define logarithmic K-theoretic localized intersection product

$$[[\ , X]] : G(X_K \times_K X_K) \longrightarrow G(X_s)$$

with the log diagonal map $X \to (X \times_S X)^\sim$ in Definition 3.18 and Proposition 3.20. It is defined as the difference of the classes of higher $\mathcal{T}or$-sheaves of even degree and odd degree. We show in Corollary 3.19.1 that the image of the logarithmic self-intersection class $(\Delta_X, \Delta_X)_S^{\log}$ in $Gr_0^F G(X_s)$ is the self-intersection product $[[\Delta_{X_K}, X]]$. The log version, Theorem 1.15, is the special case of Theorem 3.25 where Γ is the diagonal Δ_{X_K}. To prove Theorem 3.25, we take an alteration $W \to X$ where W is a projective and strictly semi-stable scheme over the integer ring $T = \mathrm{Spec}\ O_L$ of a finite normal totally ramified extension L of K. By the definition of the Swan conductor, it is reduced to an equality

$$q \sum_{\sigma \in P_{L/K}} \mathrm{sw}(\sigma) \mathrm{Tr}\ (\Gamma_\sigma^* \circ \sigma_* : H^*(W_{\bar L}, \mathbf{Q}_\ell)) = -[W : X] \deg_{X_s}[[\Gamma, X]]$$

where Γ_σ denotes the pull-back of Γ by $W_L \times_L W_{\sigma,L} \to X_K \times_K X_K$ and W_σ denotes the conjugate of W by σ and q is the inseparable degree of L over K. By using the associativity, Proposition 2.23, of the localized intersection product and using an interpretation, Corollary 2.22.2, of the Swan character as the localized intersection product, we show an equality

$$-[W : X] \deg_{X_s}[[\Gamma, X]] = q \sum_{\sigma \in P_{L/K}} \mathrm{sw}(\sigma) \cdot \deg_{W_t} \Delta^*_{W_t}(\Gamma_{\sigma,t}).$$

in Lemma 4.6. In the right hand side, $\Gamma_{\sigma,t} \in G((W \times_T W)^{\sim}_t)$ denotes the reduction of Γ_σ and $\Delta^*_{W_t} : G((W \times_T W)^{\sim}_t) \to G(W_t)$ denotes the pull-back by the log diagonal map. The proof is completed by substituting the log Lefschetz trace formula, Theorem 3.39,

$$\mathrm{Tr}\ (\Gamma^*_\sigma \circ \sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell)) = \deg_{W_t} \Delta^*_{W_t}(\Gamma_{\sigma,t}).$$

The proof outlined above is compared to Bloch's original proof in [3] as follows. In the original proof, the main steps are the following.

1'. Computation of the Euler characteristic of the closed fiber.

2'. Projection formula for localized intersection product.

3'. Computation of the trace on etale cohomology.

Each of items 1'-3' corresponds to each of items 1-3 above, respectively. In the original proof, the step 1' is carried out by a detailed combinatorial analysis peculiar to the intersection product on surfaces. In this paper, by introducing the log version, we avoid the difficulty in this step. The idea is that putting log structure on boundary is virtually equivalent to cutting off the boundary, the closed fiber in our case. A prototype of this philosophy is the Lefschetz trace formula for an open variety, Lemma 1.21. In this paper, it is realized as Proposition 3.20 which asserts that the logarithmic localized intersection product is in fact factored through the generic fiber. Non-logarithmic localized intersection product need not share this property. The step 2' is generalized to the theory of localized intersection product using K-theory. An advantage of the use of K-theory lies in that a crucial associativity formula, Proposition 2.23.1, is derived from the associativity of derived tensor product. Finally, the log Lefschetz trace formula, Theorem 3.39, facilitates the computation in the step 3' in higher dimension.

The idea behind the definition of the localized intersection product is as follows. If X is a smooth scheme over a field F, the intersection product of cycles V and W on X is defined to be the pull-back of $V \times W$ in $X \times_F X$ by the diagonal embedding $X \to X \times_F X$. Our aim is to generalize it to a regular flat scheme X over a discrete valuation ring O_K. The difficulty here is that, contrary to the case over a field, the immersion $X \to X \times_{O_K} X$ is not a regular immersion unless X is smooth over O_K. If we had a base field F of O_K, the fiber product $X \times_{O_K} X$ should be a divisor of a regular scheme $X \times_F X$. If D is a divisor of a regular scheme P, one can almost compute the intersection product of cycles on D with respect to P purely in terms of $\mathcal{T}or$-sheaves on D, as in Proposition 2.14. Although the product $X \times_{O_K} X$ may not be globally a divisor of a regular scheme, we can make a suitable definition of product using $\mathcal{T}or$-sheaves, based on the fact that it is locally a divisor of a smooth scheme over X. The product thus defined is in fact supported in the non-smooth locus of X and deserves to be called the localized intersection product. A relation with the localized intersection product in the setting of Chow groups defined by Abbés in [1] is given in Corollary 2.20.

In the classical case, the Lefschetz trace formula is a rather formal consequence of Poincaré duality, Künneth formula, the cycle map and the compatibility of the trace map with the degree map. For log etale cohomology, Poincaré duality and Künneth formula are already established in [23]. We consider the chern character map to log etale cohomology in place of the cycle map. The required compatibility is reduced to that for the usual etale cohomology.

The content of each section is as follows. In Section 1, we state the main result, Theorem 1.10, and its log version, Theorem 1.15, and prove their equivalence. In Section 2, we develop generality on localized K-theoretic intersection product. In Section 3, we define logarithmic intersection product, using the logarithmic diagonal map and the theory established in Section 2. We formulate Theorem 3.25, which contains Theorem 1.15 as a special case, in terms of logarithmic intersection product. We also state and prove logarithmic Lefschetz trace formula, Theorem 3.39. In the last Section 4, we prove Theorem 3.25 and thus complete the proof of Theorems 1.10 and 1.15.

In this paper, we use many of results in the paper [30]. However, Definition (1.1), proof of Proposition (3.1), and Proposition (4.1) in [30] are not correct. Definition (1.1) is corrected in [31] Section 4 Definition 1 and is also copied in this paper as Definition 2.32. Proposition (3.1) is reproved as Lemma 1.25. A corrected statement of Proposition (4.1) and its proof is given in §2.5 Corollary 2.42. The author of [30] apologize for the mistakes and inconvenience.

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Table of contents.

1. Conductor formula.

In this section, we recall the precise formulation of the conductor formula and give the exact statement of the main results. We recall the definition of conductor and localized chern classes in §§1.1 and 1.2 respectively. We state the main results, Theorem 1.10 and its log version Theorem 1.15 in §1.3. We show a weaker version Corollary 1.11 of Theorem 1.10 is equivalent to Theorem 1.15 in §1.4. After recalling cotangent complexes in §1.5, we show that Theorem 1.10 is equivalent to its Corollary 1.11 in §1.6.

1.1. Artin and Swan conductors.

We recall generalities on conductor. Basic references are [27] Chapitres IV, VI and [28] Partie III §3.4.

Let K be a discrete valuation field with perfect residue field F. Let ℓ be a prime number different from the characteristic p of F and $G_K \to GL_{\mathbf{Q}_\ell}(V)$ be a continuous ℓ-adic representation of the absolute Galois group $G_K = \mathrm{Gal}(\bar K/K)$. We recall the definition of the Artin conductor $\mathrm{Art}(V)$ and the Swan conductor $\mathrm{Sw}(V)$ of V.

Let L be a finite Galois extension of K of Galois group $G_{L/K}$. Assume the integer ring O_L is a discrete valuation ring and let E be the residue field of L. The Artin character $\mathrm{a}_{L/K}$ and the Swan character $\mathrm{sw}_{L/K}$ of $G_{L/K}$ are defined by

$$\mathrm{a}_{L/K}(\sigma) = \begin{cases} \mathrm{length}_{O_K}\Omega^1_{O_L/O_K} & \text{if } \sigma = 1,\\ -\mathrm{length}_{O_K} O_L/(\sigma(x) - x : x \in O_L) & \text{if } \sigma \neq 1, \end{cases}$$

$$\mathrm{sw}_{L/K}(\sigma) = \begin{cases} \mathrm{length}_{O_K}\Omega^1_{O_L/O_K} - ([L:K] - [E:F]) & \text{if } \sigma = 1,\\ -\mathrm{length}_{O_K} O_L/(\frac{\sigma(x)}{x} - 1 : x \in L^\times) & \text{if } \sigma \neq 1 \end{cases}$$

for $\sigma \in G_{L/K}$. We call the p-Sylow subgroup $P_{L/K}$ of the inertia subgroup $I_{L/K}$ of $G_{L/K}$ the wild inertia subgroup. If $\sigma \in I_{L/K}$ and π is a prime element of L, the ideals $(\sigma(x) - x, x \in O_L)$ and $(\sigma(x)/x - 1 : x \in L^\times)$ are generated by $\sigma(\pi) - \pi$ and by $\sigma(\pi)/\pi - 1$ respectively. Hence we have $\mathrm{a}(\sigma) = -\mathrm{ord}_L(\sigma(\pi) - \pi)$ and $\mathrm{sw}(\sigma) = -\mathrm{ord}_L(\sigma(\pi)/\pi - 1)$ for $\sigma \neq 1, \in I_{L/K}$. For $\sigma \in G_{L/K}$, the condition $-\mathrm{sw}_{L/K}(\sigma) > 0$ is equivalent to $\sigma \in P_{L/K} - \{1\}$ and the condition $-\mathrm{a}_{L/K}(\sigma) > 0$ is equivalent to $\sigma \in I_{L/K} - \{1\}$.

Let K' be the henselization of K. We fix an embedding $K' \to \bar K$ and identify the absolute Galois group $G_{K'}$ with a decomposition subgroup $D_K \subset G_K$. Let $I_K = \mathrm{Gal}(\bar K/K'^{\mathrm{ur}}) \subset G_K$ be the inertia group of K corresponding to the maximum unramified extension K'^{ur} of K'. We call the pro-p Sylow subgroup $P_K = \mathrm{Gal}(\bar K/K'^{\mathrm{tr}}) \subset I_K$ the wild inertia group of K. It corresponds to the maximum tamely ramified extension $K'^{\mathrm{tr}} = K'^{\mathrm{ur}}(\pi^{1/m}; p \not| m)$ of K' where π is a prime element of K.

Let $G_K \to GL_{\mathbf{Q}_\ell}(V)$ be an ℓ-adic representation. The image of the wild inertia P_K is finite. Let L be a finite Galois extension of the henselization K' such that $P_L = P_K \cap \mathrm{Gal}(\bar K/L)$ acts trivially on V. We identify $P_{L/K'} = P_K/P_L$ as a subgroup of the Galois group $G_{L/K'}$. An action of $P_{L/K'}$ on V is well-defined by the assumption on L. The Swan conductor $\mathrm{Sw}(V)$ is defined as an intertwining number

$$\mathrm{Sw}(V) = \frac{1}{[L:K']} \sum_{\sigma \in P_{L/K'}} \mathrm{sw}_{L/K'}(\sigma)\mathrm{Tr}(\sigma : V).$$

Note that $\mathrm{sw}_{L/K'}(\sigma) = 0$ unless $\sigma \in P_{L/K'}$ and the sum is taken over the subgroup $P_{L/K'} \subset G_{L/K'}$. It is a theorem that $\mathrm{Sw}(V)$ is a non-negative integer. It is 0 if and only if the action of P_K is trivial. The Artin conductor is defined by the equality $\mathrm{Art}(V) = \dim V - \dim V^I + \mathrm{Sw}(V)$ where V^I denotes the I-fixed part. The fact that the right hand side is independent of the choice of L is a consequence of the following Lemma.

Lemma 1.1 ([27] Chapitre IV Proposition 3) *Let $M \supset L \supset K$ be finite Galois extensions of a henselian discrete valuation field K. Then for an element $\sigma \in G_{L/K}$, we have*

$$[M : L]\mathrm{a}_{L/K}(\sigma) = \sum_{\tau \mapsto \sigma} \mathrm{a}_{M/K}(\tau) \quad \text{and} \quad [M : L]\mathrm{sw}_{L/K}(\sigma) = \sum_{\tau \mapsto \sigma} \mathrm{sw}_{M/K}(\tau).$$

We give a proof of Lemma 1.1 as an application of the localized K-theoretic intersection product in §3.4 after Corollary 3.34.

For an endomorphism $f : V \to V$ of an ℓ-adic representation of G_K, we define the Swan conductor $\mathrm{Sw}(f : V)$ as follows. Take a finite Galois extension L of the henselization K' such that P_L acts trivially on V as above. Then we put

$$\mathrm{Sw}(f : V) = \frac{1}{[L : K']} \sum_{\sigma \in P_{L/K'}} \mathrm{sw}_{L/K'}(\sigma)\mathrm{Tr}(f \circ \sigma : V).$$

It also follows from Lemma 1.1 that the right hand side is independent of the choice of L.

Let $K_1 \supset K$ be a discrete valuation field with perfect residue field. Assume that the valuation of K_1 is an extension of that of K and a prime element of K is a prime element of K_1. Then it follows immediately from the definition that the conductors $\mathrm{Art}(V), \mathrm{Sw}(V)$ and $\mathrm{Sw}(f : V)$ are equal to those of the pull-back to G_{K_1}.

We give a geometric interpretation of the Artin and Swan characters which plays a crucial role in the proof of the conductor formula. For simplicity, we assume L is a totally ramified finite separable extension of a discrete valuation field K with perfect residue field. We introduce some notations. We put

$$\Omega^1_{O_L/O_K}(\log/\log) = (\Omega^1_{O_L/O_K} \oplus (O_L \otimes_{\mathbf{Z}} L^\times))/(da - a \otimes a : a \in O_L - \{0\}, 1 \otimes b : b \in K^\times).$$

For an element $a \in L^\times$, we put $d\log a = 1 \otimes a$. We put $S = \mathrm{Spec}\ O_K$ and $T = \mathrm{Spec}\ O_L$. Let $(T \times_S T)^\sim$ denote the blow-up of $T \times_S T = \mathrm{Spec}\ O_L \otimes_{O_K} O_L$ at the closed point. Let $T = \Delta \to T \times_S T$ be the diagonal map and $T = \tilde{\Delta} \to (T \times_S T)^\sim$ be the induced map. We assume further L is a Galois extension. For $\sigma \in G_{L/K}$, let $T = T_\sigma \to T \times_S T$ be the graph of $\sigma : T \to T$. It is defined by the surjection $O_L \otimes_{O_K} O_L \to O_L : a \otimes b \mapsto a\sigma(b)$. Let $T = \tilde{T}_\sigma \to (T \times_S T)^\sim$ be the immersion induced by the graph of σ. If $\sigma = 1$, we have $T_1 = \Delta$ and $\tilde{T}_1 = \tilde{\Delta}$.

Lemma 1.2 *Let L be a totally ramified finite separable extension of K of degree l.*

1. The kernel and the cokernel of $\Omega^1_{O_L/O_K} \to \Omega^1_{O_L/O_K}(\log/\log)$ are isomorphic respectively to the kernel and the cokernel of $O_L/m_K O_L \to O_L/m_L : 1 \mapsto l$.

2. The O_L-module $\Omega^1_{O_L/O_K}$ is isomorphic to the conormal sheaf $N_{T/T\times_S T} = \mathcal{T}or_1^{O_{T\times_S T}}(O_T, O_T)$ and $\Omega^1_{O_L/O_K}(\log/\log)$ is isomorphic to $N_{T/(T\times_S T)^\sim} = \mathcal{T}or_1^{O_{(T\times_S T)^\sim}}(O_T, O_T)$.

3. Assume L is a Galois extension of group $G_{L/K}$. Then we have

$$\mathrm{a}_{L/K}(\sigma) = \begin{cases} \mathrm{length}_{O_K}\Omega^1_{O_L/O_K} = \mathrm{length}_{O_K}\mathcal{T}or_1^{O_{T\times_S T}}(O_T, O_T) & \text{if } \sigma = 1, \\ -\mathrm{length}_{O_K} O_T \otimes_{O_{T\times_S T}} O_{T_\sigma} & \text{if } \sigma \neq 1, \end{cases}$$

$$\mathrm{sw}_{L/K}(\sigma) = \begin{cases} \mathrm{length}_{O_K}\Omega^1_{O_L/O_K}(\log/\log) = \mathrm{length}_{O_K}\mathcal{T}or_1^{O_{(T\times_S T)^\sim}}(O_T, O_T) & \text{if } \sigma = 1, \\ -\mathrm{length}_{O_K} O_T \otimes_{O_{(T\times_S T)^\sim}} O_{T_\sigma} & \text{if } \sigma \neq 1. \end{cases}$$

4. For $\sigma \in G_{L/K}$, the closed point of $\tilde{T}_\sigma \subset (T\times_S T)^\sim$ is the same as that of $T = \tilde{T}_1 \subset (T\times_S T)^\sim$ if and only if $\sigma \in P_{L/K}$. Consequently, if $\sigma \notin P_{L/K}$, the intersection $\tilde{T}_\sigma \cap \tilde{T}_1$ is empty.

Here we give proofs based on explicit computations. More conceptual proofs for 1 and 2 will be given later in Lemma 1.19.3 and Corollary 3.19.

Proof. 1. We put

$$\Omega^1_{O_L/O_K}(\log) = (\Omega^1_{O_L/O_K} \oplus (O_L \otimes_{\mathbf{Z}} L^\times))/(da - a\otimes a : a \in O_L - \{0\}).$$

Let π' be a prime element of L and $f(x) \in O_K[x]$ be the minimal polynomial of π'. Then we have $\Omega^1_{O_L/O_K} = O_L/(f'(\pi'))\cdot d\pi'$ and $\Omega^1_{O_L/O_K}(\log) = O_L/(\pi'\cdot f'(\pi'))\cdot d\log\pi'$. Hence the canonical map $\Omega^1_{O_L/O_K} \to \Omega^1_{O_L/O_K}(\log)$ is injective and the cokernel is isomorphic to O_L/m_L. The polynomial $f(x)$ is an Eisenstein polynomial and $\pi = -f(0)$ is a prime element of K. We show that the map $O_L \to \Omega^1_{O_L/O_K}(\log)$ sending 1 to $d\log\pi$ induces an injection $O_L/m_K O_L \to \Omega^1_{O_L/O_K}(\log)$. Since $\pi d\log\pi = d\pi = 0$, it induces a map $O_L/m_K O_L \to \Omega^1_{O_L/O_K}(\log)$. We show the injectivity. We define a polynomial $h(x) \in O_K[x]$ by $f(x) = xh(x) - \pi$. Since $\pi = \pi' h(\pi')$, we have

$$\begin{aligned}\pi/\pi'\cdot d\log\pi &= h(\pi')(d\log h(\pi') + d\log\pi') = dh(\pi') + h(\pi')d\log\pi' \\ &= (h'(\pi')\pi' + h(\pi'))d\log\pi' = f'(\pi')d\log\pi' \neq 0\end{aligned}$$

in $\Omega^1_{O_L/O_K}(\log)$. Thus the injectivity is proved. Since $\Omega^1_{O_L/O_K}(\log/\log)$ is the cokernel of this map, the assertion follows easily from that the composition map $O_L/m_K O_L \to \Omega^1_{O_L/O_K}(\log) \to O_L/m_L$ sends 1 to l.

2. The assertion for $\Omega^1_{O_L/O_K}$ is well-known. We show the assertion for $\Omega^1_{O_L/O_K}(\log/\log)$. We give a non-canonical description of $(T\times_S T)^\sim$. Let $f \in O_K[x]$ be the minimal polynomial of π' over K as above. Since it is an Eisenstein polynomial, the polynomial $g(x) = f(\pi' x)/\pi'^l$ is in $O_L[x]$. We have $T\times_S T \simeq \mathrm{Spec} O_L[x]/(f)$ and $(T\times_S T)^\sim \simeq \mathrm{Spec} O_L[x]/(g)$. The diagonal map $T \to (T\times_S T)^\sim$ is defined by $x \mapsto 1$. Hence the conormal sheaf $N_{T/(T\times_S T)^\sim}$ is identified with $O_L/(g'(1)) = O_L/(f'(\pi')/\pi'^{l-1})$. On the other hand, by the equality $\pi/\pi'\cdot d\log\pi = f'(\pi')d\log\pi'$ proved in 1, we see that $\Omega^1_{O_L/O_K}(\log/\log)$ is isomorphic to $O_L/(\pi' f'(\pi')/\pi)$. Thus the assertion follows.

3. The equalities for $\sigma = 1$ follow from 1 and 2. We assume $\sigma \neq 1$. In the explicit description above, T_σ is defined by the map $O_L[x]/(f) \to O_L : x \mapsto \sigma(\pi')$ and $\tilde{T}_\sigma$ is defined by the

map $O_L[x]/(g) \to O_L : x \mapsto \sigma(\pi')/\pi'$. The O_T-modules corresponding to $O_L/(\sigma(\pi') - \pi')$ and $O_L/(\sigma(\pi')/\pi' - 1)$ are identified with $O_T \otimes_{O_{T\times_S T}} O_{T_\sigma}$ and $O_T \otimes_{O_{(T\times_S T)^\sim}} O_{T_\sigma}$ respectively.

4. Since $-\mathrm{sw}_{L/K}(\sigma) > 0$ is equivalent to $\sigma \in P_{L/K} - \{1\}$, it follows from 3.

1.2. Localized Chern class.

We recall the definition and basic properties of localized chern classes. Basic references are [8] Chapters 18 and 20 and [3] Section 1. In this subsection, S denotes a regular noetherian scheme. The dimension $\dim S$ is defined to be the supremum of the dimensions of the local rings $\dim O_{S,s}$. In the following, we assume S is of finite equi-dimension. For a point s of S, we put $\dim s = \dim S - \dim O_{S,s}$. In this subsection, X denotes a scheme of finite type over S and $f : X \to S$ denotes the structural map. We put $\dim x = \mathrm{tr.deg}_{\kappa(f(x))}\kappa(x) + \dim f(x)$ for $x \in X$. For an integer $i \geq 0$, let X_i denote the set $\{x \in X | \dim x = i\}$. For a closed subset $V \subset X$, we put $\dim V = \sup_{x\in V} \dim x$. Note that the function dim depends on the base scheme S.

For a scheme X of finite type over a regular noetherian base scheme S as above and for an integer $i \geq 0$, the Chow group $CH_i(X)$ is defined as the cokernel $\bigoplus_{y\in X_{i+1}} \kappa(y)^\times \xrightarrow{d} \bigoplus_{x\in X_i} \mathbf{Z}$. The (x, y)-component $d_{x,y} : \kappa(y)^\times \to \mathbf{Z}$ of d is characterized as follows. Let Y be the closure of $\{y\}$ with the reduced subscheme structure. If $x \in Y$, the map $d_{x,y}$ satisfies $d_{x,y}f = \mathrm{length} O_{Y,x}/(f)$ for $f \in O_{Y,x}, \neq 0$ and, if $x \notin Y$, it is the 0-map. For a closed subscheme $Z \subset X$ and an integer $i \geq 0$, an element of the bivariant Chow cohomology group $CH^i(Z \to X)$ is a collection of maps $CH_j(X') \to CH_{j-i}(Z \times_X X')$ for schemes X' of finite type over X and for integers $j \geq i$, satisfying certain natrual functorial properties.

Let $\mathcal{K} = (\mathcal{K}_q, d_q)_q$ be a bounded chain complex of locally free O_X-modules of finite ranks. Assume that on the complement $U = X - Z$, the restriction $\mathcal{K}|_U$ is acyclic except at degree 0 and the cohomology sheaf $\mathcal{H}_0(\mathcal{K})|_U$ is locally free of rank $n - 1$. Then for $i \geq n$, the localized Chern class ${c_i}^X_Z(\mathcal{K}) \in CH^i(Z \to X)$ is defined. We regard the total localized chern class $c^X_Z(\mathcal{K}) = ((c_i(\mathcal{K}))_{i<n}, ({c_i}^X_Z(\mathcal{K}))_{i\geq n})$ as an invertible element in the ring $\prod_{i<n} CH^i(X \to X) \times \prod_{i\geq n} CH^i(Z \to X)$. The localized Chern classes satisfy the following properties.

Proposition 1.3 ([3] Proposition (1.1)) *Let Z be a closed subscheme of X and $\mathcal{K}$ be a finite chain complex of locally free O_X-modules of finite ranks. Assume that on the complement $U = X - Z$, the restriction $\mathcal{K}|_U$ is acyclic except at degree 0 and the cohomology sheaf $\mathcal{H}_0(\mathcal{K})|_U$ is locally free of rank $n - 1$.*

1. If $Z = X$, we have $c^X_Z(\mathcal{K}) = \prod_q c(\mathcal{K}_q)^{(-1)^q}$.

2. For a quasi-isomorphism $\mathcal{K} \to \mathcal{K}'$, we have $c^X_Z(\mathcal{K}) = c^X_Z(\mathcal{K}')$.

3. Let $\mathcal{E}$ be a locally free O_X-module of finite rank. Then for $i \geq n$ and for an integer i', we have ${c_i}^X_Z(\mathcal{K})c_{i'}(\mathcal{E}) = c_{i'}(\mathcal{E}|_Z){c_i}^X_Z(\mathcal{K})$. Let $\mathcal{K}'$ be another finite chain complex of locally free O_X-modules of finite ranks such that the restriction $\mathcal{K}'|_U$ is acyclic except at degree 0 and the cohomology sheaf $\mathcal{H}_0(\mathcal{K}')|_U$ is locally free of rank $n' - 1$. Then for $i \geq n, i' \geq n'$, we have ${c_i}^X_Z(\mathcal{K})c_{i'}(\mathcal{K}') = {c_{i'}}^X_Z(\mathcal{K}')c_i(\mathcal{K})$.

4. Let $\mathcal{K}'$ and $\mathcal{K}''$ be finite chain complexes of locally free O_X-modules of finite ranks such that the restriction $\mathcal{K}'|_U$ and $\mathcal{K}''|_U$ are acyclic except at degree 0 and the cohomology sheaves $\mathcal{H}_0(\mathcal{K}')|_U$ and $\mathcal{H}_0(\mathcal{K}'')|_U$ are locally free of rank $n' - 1$ and $n'' - 1$ respectively and let $\mathcal{K}' \to \mathcal{K} \to \mathcal{K}'' \to$ be a *distinguished triangle. Then for an integer $i \geq n = n' + n'' - 1$, we have*

$$c_{i}{}_{Z}^{X}(\mathcal{K}) = \sum_{i'+i''=i, i'\geq n'} c_{i'}{}_{Z}^{X}(\mathcal{K}')c_{i''}(\mathcal{K}'') + \sum_{i'+i''=i, i''\geq n''+(i-n)} c_{i''}{}_{Z}^{X}(\mathcal{K}'')c_{i'}(\mathcal{K}').$$

5. Let $Z \overset{i}{\subset} Z' \subset X$ be closed immersions. Let i_* denote the collection of the induced maps $i_* : CH_*(Z \times_X X') \to CH_*(Z' \times_X X')$ for schemes X' of finite type over X. Then we have $i_* \circ c_Z^X(\mathcal{K}) = c_{Z'}^X(\mathcal{K})$.

Let $f : X'' \to X'$ be a morphism of finite type over X and let Z' and Z'' be the inverse images of Z.

6. Assume f is proper and let $f_* : CH_*(X'') \to CH_*(X')$ and $f_{Z'*} : CH_*(Z'') \to CH_*(Z')$ be the induced maps. Then we have $c_Z^X(\mathcal{K}) \circ f_* = f_{Z''*} \circ c_Z^X(\mathcal{K})$.

7. Assume f is flat and let $f^* : CH_*(X') \to CH_*(X'')$ and $f_Z^* : CH_*(Z') \to CH_*(Z'')$ be the induced maps. Then we have $f_{Z''}^* \circ c_Z^X(\mathcal{K}) = c_Z^X(\mathcal{K}) \circ f^*$.

Let $\mathcal{F}$ be an O_X-module such that the restriction $\mathcal{F}|_U$ is locally free of rank n. If $\mathcal{F}$ has a finite resolution $\mathcal{E}_\bullet \to \mathcal{F}$ by locally free O_X-modules $\mathcal{E}_q$ of finite rank, the localized chern class $c_{i}{}_{Z}^{X}(\mathcal{F})$ for $i > n$ is defined as $c_{i}{}_{Z}^{X}(\mathcal{E}_\bullet)$. By Proposition 1.3.2, it is independent of the choice of a resolution. If X is separated, regular and noetherian, any coherent sheaf has a finite resolution by locally free O_X-modules.

Lemma 1.4 ([13] Corollary 2.2.7.1) *Let X be a separated regular noetherian scheme of dimension n and $\mathcal{F}$ be a coherent O_X-module. Then there exists a resolution $0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_0 \to \mathcal{F} \to 0$ of $\mathcal{F}$ by locally free O_X-modules of finite rank.*

For a locally free sheaf on a divisor, its localized chern class is computed as a special case of Riemann-Roch without denominator as follows.

Lemma 1.5 (cf. [8] Theorem 15.3) *Let D be a Cartier divisor of a scheme X and $i : D \to X$ be the immersion. Let $\mathcal{E}$ be a locally free O_D-module of rank n. Put*

$$a_j(\mathcal{E}) = \sum_{k=j}^{n} \binom{k}{j} c_{n-k}(\mathcal{E}(-D)), \qquad b_j(\mathcal{E}) = (-1)^j \sum_{k=j}^{n} \binom{k}{j} c_{n-k}(\mathcal{E}) \in CH^*(D \to D)$$

so that we have $\sum_{j=0}^{n} a_j(\mathcal{E})t^j = \sum_{k=0}^{n}(1+t)^k c_{n-k}(\mathcal{E}(-D)), \sum_{j=0}^{n} b_j(\mathcal{E})t^j = \sum_{k=0}^{n}(1-t)^k c_{n-k}(\mathcal{E})$ in $CH^*(D \to D)[t]$. Assume there exists a locally free O_X-module $\tilde{\mathcal{E}}$ of finite rank and a surjection $\tilde{\mathcal{E}} \to i_*\mathcal{E}$ so that the localized chern class $c_D^X(i_*\mathcal{E})$ is defined as an invertible element of the ring $CH^0(X \to X) \times \prod_{j>0} CH^j(D \to X)$. Then we have equalities

$$(c_D^X(i_*\mathcal{E}) - 1) \cap [X] = (\sum_{j=1}^{n} a_j(\mathcal{E}) D^{j-1}) c(\mathcal{E}(-D))^{-1} \cap [D],$$

$$(c_D^X(i_*\mathcal{E})^{-1} - 1) \cap [X] = (\sum_{j=1}^{n} b_j(\mathcal{E}) D^{j-1}) c(\mathcal{E})^{-1} \cap [D]$$

in $CH_*(D)$.

Sketch of Proof. By deforming to normal bundle, we may assume X is a $\mathbf{P}^1$-bundle over D and the immersion $i : D \to X$ is a section. Then $\mathcal{E}$ is the restriction to D of the pull-back $\mathcal{E}_X$ of $\mathcal{E}$ to X. Since the map $i_* : CH_*(D) \to CH_*(X)$ is injective, it is reduced to the equality for the usual chern class $c(i_*\mathcal{E})$. By the locally free resolution $0 \to \mathcal{E}_X(-D) \to \mathcal{E}_X \to i_*\mathcal{E} \to 0$, we have

$$c(i_*\mathcal{E}) - 1 = c(\mathcal{E}_X(-D))^{-1}(c(\mathcal{E}_X) - c(\mathcal{E}_X(-D))) = c(\mathcal{E}_X(-D))^{-1}(\sum_{j=0}^{n} a_j(\mathcal{E})D^j - a_0(\mathcal{E})).$$

The second equality is proved similarly.

Corollary 1.6 *If $n = 1$, we have*

$$(c_D^X(\mathcal{E}) - 1) \cap [X] = c(\mathcal{E}(-D))^{-1} \cap [D] \qquad \text{and} \qquad (c_D^X(\mathcal{E})^{-1} - 1) \cap [X] = -c(\mathcal{E})^{-1} \cap [D].$$

In the most applications in this paper, it is sufficient to compute the localized chern class in the following case.

Lemma 1.7 *Let $f : \mathcal{L} \to \mathcal{E}$ be a morphism of locally free O_X-modules and $n \geq 0$ be an integer. Assume that $\mathcal{F} = \mathrm{Coker}\ f$ is locally generated by n sections and that $\mathrm{rank}\ \mathcal{E} - \mathrm{rank}\ \mathcal{L} = n - 1$. Let $\mathcal{K}$ be the complex $[\mathcal{L} \xrightarrow{f} \mathcal{E}]$ of length 1 where $\mathcal{E}$ is put on degree 0, let $i : Z \to X$ be the closed immersion defined by the annihilater ideal $\mathrm{Ann}\ \Lambda^n\mathcal{F}$ and let $\mathcal{L}_Z$ be the O_Z-module $L^1 i^*\mathcal{K}$. Then,*

1. The restriction $\mathcal{F}|_{X-Z}$ is locally free of rank $n-1$ and the localized chern class $c_n{}_Z^X(\mathcal{K}) \in CH^n(Z \to X)$ is defined. The O_Z-module $\mathcal{F}|_Z = L^0 i^*\mathcal{K}$ is locally free of rank n and the O_Z-module $\mathcal{L}_Z = L^1 i^*\mathcal{K}$ is invertible.

Let $\varphi : W \to X$ be a morphism of schemes. Assume that W is of finite type over a regular noetherian scheme and is of dimension p.

2. Assume W is a scheme over Z and define O_W-modules $\mathcal{E}_W$ and $\mathcal{L}_W$ by $\mathcal{E}_W = L^0\varphi^*\mathcal{K}$ and $\mathcal{L}_W = L^1\varphi^*\mathcal{K}$. Then we have $\mathcal{E}_W = \varphi^*\mathcal{F}$ and $\mathcal{L}_W = \varphi^*\mathcal{L}_Z$. The O_W-module $\mathcal{E}_W$ is locally free of rank n and the O_W-module $\mathcal{L}_W$ is invertible. We have

$$c_n{}_Z^X(\mathcal{K}) \cap [W] = c_n(\mathcal{E}_W \otimes \mathcal{L}_W^{\otimes -1}) \cap [W] \quad \text{in } CH_{p-n}(W).$$

3. Assume $D = Z \times_X W$ is a Cartier divisor of W. Let $\varphi_D : D \to Z$ be the restriction and let $\mathcal{L}_D$ be the invertible O_D-module $L^1(i \circ \varphi_D)^*\mathcal{K} = \varphi_D^*\mathcal{L}_Z$. Then, the O_W-module $L^0\varphi^*\mathcal{K} = \varphi^*\mathcal{F}$ is an extension of a locally free O_W-module $\mathcal{E}'_W$ of rank $n-1$ by an invertible O_D-module $\mathcal{L}_D \otimes_{O_W} O_W(D)$. We have an equality

$$c_n{}_Z^X(\mathcal{K}) \cap [W] = c_{n-1}(\mathcal{E}'_W|_D \otimes \mathcal{L}_D^{\otimes -1}) \cap [D] \quad \text{in } CH_{p-n}(D).$$

Corollary 1.8 *Let $\varphi : W \to X$ be a morphism of schemes as above. Assume that W is integral, is of finite type over a regular noetherian scheme and is of dimension p. Let $\pi : W' \to W$ be the blow-up at the inverse image $Z_W = Z \times_X W$. Let $D = Z \times_X W'$ be the exceptional divisor and $\pi_D : D \to Z_W$ be the restriction of π. Let $\mathcal{E}'_{W'}$ be the locally free quotient of the $O_{W'}$-module $(\varphi \circ \pi)^*\mathcal{F}$ and let $\mathcal{L}_D$ be the invertible O_D-module $L^1(\varphi \circ \pi_D)^*\mathcal{K} = (\varphi \circ \pi_D)^*\mathcal{L}_Z$. Then, we have*

$$c_n{}_Z^X(\mathcal{K}) \cap [W] = \pi_{D*}(c_{n-1}(\mathcal{E}'_{W'}|_D \otimes \mathcal{L}_D^{\otimes -1}) \cap [D]) \quad \text{in } CH_{p-n}(Z_W).$$

Proof of Lemma 1.7. 1. The assertion is local on X. The assumption means that, locally on X, there exists a quasi-isomorphism $\mathcal{K} \to [O_X \xrightarrow{f} O_X^n]$. If the map f is defined by a vector $(a_1, \ldots, a_n) \in O_X^n$, the subscheme Z is defined by the ideal $(a_1, \ldots, a_n)$. The assertion follows from this immediately.

2. The first assertion is clear from the local description above. The localized chern class ${c_n}_Z^X(\mathcal{K}) \cap [W]$ is equal to the degree n part of the total chern class $c(\mathcal{E}_W)c(\mathcal{L}_W)^{-1} \cap [W]$. It is equal to $\sum_{p+q=n}(-1)^q c_p(\mathcal{E}_W)c_1(\mathcal{L}_W)^q \cap [W]$ and further to the right hand side.

3. Since ${c_n}_Z^X(\mathcal{K}) \cap [W] = {c_n}_D^W(L\varphi^*\mathcal{K}) \cap [W]$, we may assume $X = W$ and $Z = D$ is a Cartier divisor of X. It means that, in the local description above, we may assume a_1 is a non-zero divisor and $a_2 = \ldots = a_n = 0$. It is clear from this that $\mathcal{K}$ is quasi-isomorphic to $\mathcal{F}[0]$ and $\mathcal{F}$ is an extension of a locally free O_X-module $\mathcal{E}'$ of rank $n-1$ by an invertible O_D-module $\mathcal{L}'_D = Hom_{O_X}(O_D, \mathcal{F})$. We have a canonical isomorphism $\mathcal{L}_D = \mathcal{T}or_1^{O_X}(O_D, \mathcal{F}) \to \mathcal{L}'_D \otimes_{O_X} O_X(-D)$. Hence $\mathcal{L}'_D$ is identified with $\mathcal{L}_D \otimes_{O_X} O_X(D)$. By the exact sequence $0 \to \mathcal{L}'_D \to \mathcal{F} \to \mathcal{E}' \to 0$, we have $c_D^X(\mathcal{K}) \cap [X] = c(\mathcal{E}')c_D^X(\mathcal{L}'_D) \cap [X]$. By the first equality of Corollary 1.6 and $\mathcal{L}'_D = \mathcal{L}_D \otimes_{O_X} O_X(D)$, we have $c_D^X(\mathcal{K}) \cap [X] = c(\mathcal{E}')([X] + c(\mathcal{L}_D)^{-1} \cap [D])$. Its degree n-part is equal to $\sum_{p+q=n-1}(-1)^q c_p(\mathcal{E}'|_D)c_1(\mathcal{L}_D)^q \cap [D]$ and further to the right hand side.

Proof of Corollary. It follows immediately from Lemma 1.7.3 and Proposition 1.3.6.

1.3. Conductor formula.

Let K be a discrete valuation field with perfect residue field F and let X be a proper flat and regular scheme over O_K purely of dimension n with smooth generic fiber. In this paper, we say that a regular scheme is of dimension n only if it is purely of dimension n. In the rest of Section 1, S will denote Spec O_K and $s =$ Spec F denotes the closed point.

We define the conductors of X. Let $d = n-1$ be the dimension of the generic fiber X_K. The Swan conductor is defined to be the alternating sum

$$\mathrm{Sw}(X_K/K) = \sum_{q=0}^{2d}(-1)^q \mathrm{Sw} H^q(X_{\bar{K}}, \mathbf{Q}_\ell).$$

The cohomology in the right hand side is the ℓ-adic etale cohomology for a prime ℓ different from the characteristic p of F. It is known that the alternating sum is independent of the choice of ℓ [25]. The Artin conductor $\mathrm{Art}(X/O_K)$ is defined by

$$\mathrm{Art}(X/O_K) = \chi(X_{\bar{K}}) - \chi(X_{\bar{F}}) + \mathrm{Sw}(X_K/K).$$

In the right hand side, χ denotes the ℓ-adic Euler number which is known to be independent of ℓ as a consequence of the Weil conjecture.

The localized self-intersection class $(\Delta_X, \Delta_X)_S \in CH_0(X_F)$ is defined as follows. The coherent O_X-module Ω^1_{X/O_K} has a locally free resolution by Lemma 1.4. Its restriction to the generic fiber is locally free of rank $n-1$. Hence the localized Chern class ${c_n}_{X_F}^X(\Omega^1_{X/O_K}) \cap [X]$ is defined as an element in $CH_0(X_F)$. The localized self-intersection class $(\Delta_X, \Delta_X)_S \in CH_0(X_F)$ is defined to be $(-1)^n {c_n}_{X_F}^X(\Omega^1_{X/O_K}) \cap [X]$. We consider its image $\deg(\Delta_X, \Delta_X)_S \in \mathbf{Z}$ by the degree map $\deg : CH_0(X_F) \to CH_0(F) = \mathbf{Z}$.

Conjecture 1.9 ([3] Conjecture) *Let K be a discrete valuation field with perfect residue field F and let X be a proper flat and regular scheme over O_K with smooth generic fiber. Then we have*

$$\mathrm{Art}(X/O_K) = -\deg(\Delta_X, \Delta_X)_S.$$

We call the formula in Conjecture 1.9 the conductor formula for X. The conductor formula in the case $\dim X = 1$ is the classical conductor-discriminant formula. In the case $\dim X = 2$, it is proved by Bloch in the same paper [3]. An explicit computation of the right hand side is given in Proposition 1.16.1 below.

To state the main result, we introduce some terminology. Let X be a regular noetherian scheme and D be a divisor of X. We say a divisor D has simple normal crossings if its irreducible components are regular and if they meet transversally. More precisely, let $D_1, \ldots, D_m$ be the irreducible components. Then for any subset $I = \{i_1, \ldots, i_s\} \subset \{1, \ldots, m\}$, the intersection $D_I = \bigcap_{i\in I} D_i = D_{i_1} \times_X \cdots \times_X D_{i_s}$ is a regular subscheme of codimension s. In other words, for each $x \in X$, there exist a regular system $t_1, \ldots, t_l$ of parameters of the regular local ring $O_{X,x}$ and an integer $0 \le r \le l$ such that the divisor D is defined by $\prod_{i=1}^r t_i$ in a neighborhood of x. We say a divisor D has normal crossings if, etale locally on X, the divisor has simple normal crossings. A divisor D with normal crossings has simple normal crossing if and only if each of its irreducible component is regular.

Our first main result is the following.

Theorem 1.10 *Let K be a discrete valuation field with perfect residue field F and let X be a proper flat and regular scheme over O_K with smooth generic fiber. We assume the following condition.*

(N) *The reduced closed fiber $(X_F)_{\mathrm{red}}$ is a divisor with normal crossings.*
Then we have

$$\mathrm{Art}(X/O_K) = -\deg(\Delta_X, \Delta_X)_S.$$

As we show in Corollary 1.14 below, if the reduced closed fiber $(X_F)_{\mathrm{red}}$ has an embedded resolution in a strong sense, Conductor formula for X is true. In §1.6, we show that Theorem 1.10 is equivalent to the following weaker version.

Corollary 1.11 *Let K and X be as in Theorem 1.10 and assume the following condition.*

(S) *The reduced closed fiber $(X_F)_{\mathrm{red}}$ is a divisor with simple normal crossings.*
Then we have

$$\mathrm{Art}(X/O_K) = -\deg(\Delta_X, \Delta_X)_S.$$

The condition (S) (resp. (N)) is equivalent to the following condition. For each closed point x in the closed fiber X_s, there exist a minimal system $(t_1, \ldots, t_r)$ of generaters of the maximal ideal m_x of the local ring $O_{X,x}$ (resp. $m_{\hat{O}^{\mathrm{ur}}_{X,x}}$ of the maximum unramified extension of the completion $\hat{O}_{X,x}$), a unit $u \in O^\times_{X,x}$ (resp. $\hat{O}^{\mathrm{ur}\times}_{X,x}$) and integers $l_1, \ldots, l_n \ge 0$ such that $\pi = u\prod_i t_i^{l_i}$ for a prime element π of K.

In §1.6, we show that Theorem 1.10 is equivalent to its Corollary 1.11 by proving the following two propositions.

Proposition 1.12 *Let X be a proper flat and regular scheme over S with smooth generic fiber. Let C be a regular closed subscheme of X supported in the closed fiber X_s and $\pi : X' \to X$ be the blow-up at C. Then, the conductor formula for X is equivalent to that for X'.*

Proposition 1.13 *Let X be a regular scheme of dimension n and D be a divisor with normal crossings. Let $\bar{D}$ be the normalization of D and V_i be the closed subset $\{x \in X | \deg_x \bar{D}_x \geq n-i\}$ with the reduced closed subscheme structure. We put $X_0 = X$ and, for $0 \leq i \leq n-2$, define $X_{i+1} \to X_i$ inductively to be the blow-up at the proper transform V_i' of V_i. Then,*

1. X_i is regular, the reduced inverse image D_i of D in X_i is a divisor with normal crossing and V_i' is regular for $0 \leq i \leq n-1$.

2. D_{n-1} has simple normal crossings.

It is clear that Propositions 1.12 and 1.13 implies the equivalence of Theorem 1.10 and Corollary 1.11. Theorem 1.10 together with Proposition 1.12 has the following consequence.

Corollary 1.14 *Let X be as in Conjecture 1.9. Assume there exists a sequence of blowing-ups $X' = X_m \to \cdots \to X_0 = X$ at regular closed subschemes supported in the closed fibers such that the reduced closed fiber $X'_{F,\mathrm{red}}$ has normal crossings. Then Conjecture 1.9 is true for X.*

In particular when $\dim X = 2$, the assumption of Corollary 1.14 is satisfied and hence we obtain a new proof of Conjecture 1.9 in this case.

We formulate a logarithmic version, Theorem 1.15, of Corollary 1.11. We assume that the reduced closed fiber of X is a divisor with simple normal crossings. We define coherent O_X-modules

$$\Omega^1_{X/S}(\log) = \left(\Omega^1_{X/S} \oplus (O_X \otimes_{\mathbf{Z}} j_* O^\times_{X_K})\right) / (da - a \otimes a : a \in O_X \cap j_* O^\times_{X_K}),$$
$$\Omega^1_{X/S}(\log/\log) = \left(\Omega^1_{X/S} \oplus (O_X \otimes_{\mathbf{Z}} j_* O^\times_{X_K})\right) / (da - a \otimes a : a \in O_X \cap j_* O^\times_{X_K}, 1 \otimes b : b \in K^\times).$$

We have canonical maps $\Omega^1_{X/S} \to \Omega^1_{X/S}(\log) \to \Omega^1_{X/S}(\log/\log)$. On the generic fiber, they induce isomorphisms $\Omega^1_{X_K/K} = \Omega^1_{X/S}|_{X_K} \to \Omega^1_{X/S}(\log)|_{X_K} \to \Omega^1_{X/S}(\log/\log)|_{X_K}$. Hence, the localized chern class $c_n{}^X_{X_F}(\Omega^1_{X/S}(\log/\log)) \cap [X] \in CH_0(X_F)$ is defined. The localized log self-intersection class $(\Delta_X, \Delta_X)^{\log}_S \in CH_0(X_F)$ is defined to be $(-1)^n c_n{}^X_{X_F}(\Omega^1_{X/O_K}(\log/\log)) \cap [X]$.

Theorem 1.15 *Let K be a discrete valuation field with perfect residue field F and let X be a proper flat and regular scheme over O_K with smooth generic fiber. We assume the following condition.*

(S) *The reduced closed fiber $(X_F)_{\mathrm{red}}$ is a divisor with simple normal crossings.*

Then we have

$$\mathrm{Sw}(X_K/K) = -\deg(\Delta_X, \Delta_X)^{\log}_S.$$

In §1.4, we show that Corollary 1.11 and Theorem 1.15 are equivalent. Proof of equivalence is similar to the proof of the conductor formula in the tame case in [2]. In §3.4, we formulate a generalization, Theorem 3.25, of Theorem 1.15. Proof of Theorem 3.25 and hence the proofs of Theorems 1.10 and 1.15 are completed in §4.

The right hand sides, $(\Delta_X, \Delta_X)_S$ and $(\Delta_X, \Delta_X)^{\log}_S$, of the conductor formula and its log version can be computed as follows.

Proposition 1.16 *Let X be a regular flat and separated scheme of dimension n of finite type over* Spec O_K *with smooth generic fiber.*

1. Regard the closed subset

$$Z = \{x \in X : X \text{ is not smooth at } x \text{ over } O_K\}$$

as a closed subscheme of X defined by the annihilater ideal Ann $\Omega^n_{X/S}$. *Let $\pi : X' \to X$ be the blow-up at Z and $D = Z \times_X X'$ be the exceptional divisor. Then the pull-back $\pi^*\Omega^1_{X/S}$ is an extension of a locally free $O_{X'}$-module $\mathcal{E}'$ of rank $n-1$ by an invertible O_D-module and we have*

$$c_n{}^X_Z(\Omega^1_{X/S}) \cap [X] = \pi_*(c_{n-1}(\mathcal{E}'|_D) \cap [D]).$$

2. Assume further the reduced closed fiber $X_{s,\mathrm{red}}$ has simple normal crossings. Let $\pi : X' \to X$ be the blow-up at the closed subscheme Z of X defined by the ideal Ann $\Lambda^n\Omega^1_{X/S}(\log/\log)$ *and $D = Z \times_X X'$ be the exceptional divisor. Then the pull-back $\pi^*\Omega^1_{X/S}(\log/\log)$ is an extension of a locally free $O_{X'}$-module $\mathcal{E}'$ of rank $n-1$ by an invertible O_D-module and we have*

$$c_n{}^X_Z(\Omega^1_{X/S}(\log/\log)) \cap [X] = \pi_*(c_{n-1}(\mathcal{E}'|_D) \cap [D]).$$

Proof is given in §1.5. Another computation of $\deg(\Delta_X, \Delta_X)_S$ in terms of the torsion parts of $\Omega^q_{X/S}$ is given in [30].

Example. Let the notation be as in Proposition 1.16. Assume $x \in X$ is an isolated non-degenerate quadratic singularity of the map $X \to S$ and assume $X - \{x\}$ is smooth over S. Then $Z = \{x\}$ with reduced scheme structure, $D \simeq \mathbf{P}^{n-1}_x$ is the exceptional divisor and $\mathcal{E}'|_D$ is a quotient of O^n_D by $O_D(-1)$. Hence $c_{n-1}(\mathcal{E}'|_D) \cap [D]$ is the class $[x']$ of a $\kappa(x)$-rational point x' of D and $c_n{}^X_Z(\Omega^1_{X/S}) \cap [X] = \pi_*[x'] = [x]$.

1.4. Log version.

In this subsection, we prove the equivalence of Corollary 1.11 and Theorem 1.15. Before starting the proof, we give a local description of the sheaves $\Omega^1_{X/S}$ and $\Omega^1_{X/S}(\log/\log)$ and study their relation. We do it by taking an immersion to a smooth scheme locally on X.

Lemma 1.17 *Let X be a regular flat scheme of dimension n of finite type over O_K. Then,*

1. For a point x in the closed fiber of X, there exist an open neighborhood U of x and a regular immersion $U \to P$ of codimension 1 into a smooth scheme P of relative dimension n over O_K.

2. Assume further that the reduced closed fiber of X has simple normal crossings. For a point x in the closed fiber of X, there exist an open neighborhood U of x and a regular immersion $U \to P$ of codimension 1 into a smooth scheme P over O_K satisfying the following condition.

Let $D_1, \ldots, D_r$ be the irreducible components of the closed fiber of U. There is an etale map $P \to \mathbf{A}^n_S =$ Spec $O_K[T_1, \ldots, T_n]$ *such that the divisor D_i is defined by the image $t_i \in \Gamma(U, O_X)$ of T_i for $1 \le i \le r$. For a prime element π of K, there exist a unit $u \in \Gamma(P, O_P^\times)$ and integers $l_1, \ldots, l_r \ge 1$ such that the closed subscheme $U \to P$ is defined by $\pi - u\prod_{i=1}^r T_i^{l_i}$.*

Proof. 1. Let $t_1, \ldots, t_k \in O_{X,x}$ be a minimal system of generators of the maximal ideal m_x of the local ring $O_{X,x}$. Let $t_{k+1}, \ldots, t_{k+m} \in O_{X,x}$ be a lifting of a transcendental basis of the residue field $\kappa(x)$ over F such that $\kappa(x)$ is a finite separable extension of $F(t_{k+1},\ldots,t_{k+m})$. The sum $k+m$ is equal to the dimension n of X. We take an open neighborhood U of x and define a map $U\to \mathbf{A}^n_{O_K}=\mathrm{Spec}\ O_K[T_1,\ldots,T_n]$ by $T_i\mapsto t_i$. Then we have $\Omega^1_{U/\mathbf{A}^n_{O_K},x}=0$. By shrinking U if necessary, we may assume $\Omega^1_{U/\mathbf{A}^n_{O_K}}=0$, namely $U\to\mathbf{A}^n_{O_K}$ is neat. By [10] Corollaire (18.4.7), further shrinking U if necessary, the map $U\to\mathbf{A}^n_{O_K}$ is factorized as the composition of a closed immersion $U\to P$ and an etale morphism $P\to\mathbf{A}^n_{O_K}$. The scheme P is smooth over O_K of relative dimension n. Hence it is regular of dimension $n+1$. Therefore the immersion $U\to P$ is regular of codimension 1.

2. Let $D_1,\ldots,D_r$ be the irreducible components of the closed fiber of X containing x. Let t_i be an element of $O_{X,x}$ defining D_i at x for $1\le i\le r$. Let l_i be the multiplicity of D_i in the closed fiber X_s and put $v=\pi/\prod_{i=1}^r t_i^{l_i}\in O_{X,x}^\times$. Let $t_1,\ldots,t_k$ be a minimal system of generators of the maximal ideal m_x extending $t_1,\ldots,t_r$. We define a regular immersion $U\to P$ and an etale morphism $P\to\mathbf{A}^n_{O_K}$ as above. Shrinking U and P if necessary, we take a unit $u\in\Gamma(P,O_P^\times)$ lifting v. Then the function $f=\pi-u\prod_{i=1}^r T_i^{l_i}$ vanishes in $O_{X,x}$. Since f is not in $m^2_{P,x}$, we have $O_{X,x}=O_{P,x}/(f)$. Hence shrinking U and P if necessary, the subscheme U is defined by the equation $f=0$. Thus the condition is satisfied.

Let $U\to P$ be an immersion as in Lemma 1.17.2. Then the divisor Δ of P defined by $\prod_{i=1}^r T_i$ has relative normal crossing over S. We put $\Omega^1_{P/S}(\log)=\Omega^1_{P/S}(\log\Delta)$.

Corollary 1.18 *Let the notation be as in Lemma 1.17 and assume that the generic fiber is smooth.*

1. Let $U\to P$ be an immersion as in Lemma 1.17.1. Then we have an exact sequence

$$0\longrightarrow N_{U/P}\longrightarrow \Omega^1_{P/S}|_U\longrightarrow\Omega^1_{U/S}\longrightarrow 0.$$

The O_U-module $\Omega^1_{P/S}|_U$ is locally free of rank n and the conormal sheaf $N_{U/P}$ is invertible.

2. Assume further that the reduced closed fiber of X has simple normal crossings and let $U\to P$ be an immersion as in Lemma 1.17.2. Then we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}0&\longrightarrow&N_{U/P}&\longrightarrow&\Omega^1_{P/S}(\log)|_U&\longrightarrow&\Omega^1_{U/S}(\log)&\longrightarrow&0\\ &&\downarrow&&\|&&\downarrow&&\\ 0&\longrightarrow&N_{U/P}\otimes_{O_K}m_K^{-1}&\longrightarrow&\Omega^1_{P/S}(\log)|_U&\longrightarrow&\Omega^1_{U/S}(\log/\log)&\longrightarrow&0.\end{array}\qquad(1)$$

The O_U-module $\Omega^1_{P/S}(\log)|_U$ is locally free of rank n and $N_{U/P}$ is invertible.

Proof. 1. For the exact sequence, it is sufficient to show the injectivity of $N_{U/P}\to\Omega^1_{P/S}|_U$. Since the generic fiber is smooth, it is injective there. Since X is normal, the map is injective. The rest of assertion is clear.

2. The exactness of the upper line is proved similarly as in 1. To obtain the lower exact sequence, we show that the map $N_{U/P}\to\Omega^1_{P/S}(\log)|_U$ is extended uniquely to a map $N_{U/P}\otimes_{O_K} m_K^{-1}\to\Omega^1_{P/S}(\log)|_U$. The generator $\pi-u\prod_i T_i^{l_i}$ of $N_{U/P}$ is mapped to $d(u\prod_i T_i^{l_i})=\pi\cdot(u^{-1}du+\sum_i l_i d\log T_i)$ in $\Omega^1_{P/S}(\log)|_U$. Since it is divisible by π, the map $N_{U/P}\to\Omega^1_{P/S}(\log)|_U$ is uniquely

extended to a map $N_{U/P} \otimes_{O_K} m_K^{-1} \to \Omega^1_{P/S}(\log)|_U$ sending the generator $(\pi - u\prod_i T_i^{l_i})/\pi$ to $u^{-1}du + \sum_i l_i d\log T_i$. Since the image of $u^{-1}du + \sum_i l_i d\log T_i$ in $\Omega^1_{U/S}(\log)$ is $d\log\pi$, the lower sequence is also exact. The rest of assertion is clear.

The sheaves $\Omega^1_{X/S}, \Omega^1_{X/S}(\log/\log)$ and $\Omega^1_{X/S}(\log)$ are related to each other as follows.

Lemma 1.19 *Let X be a regular flat scheme of dimension n of finite type over O_K. Assume that the reduced closed fiber of X has simple normal crossings and that the generic fiber is smooth. Let $D_1, \ldots, D_m$ be the irreducible components of the reduced closed fiber $X_{s,\mathrm{red}}$ and l_i be the multiplicity of D_i in X_s. Then,*

1. For each irreducible component D_i, the valuation $\mathrm{ord}_{D_i} : j_*O^\times_{X_K} \to \mathbf{Z}_{D_i} \to O_{D_i}$ and the 0-map $\Omega^1_{X/S} \to O_{D_i}$ induce a map $\mathrm{res}_i : \Omega^1_{X/S}(\log) \to O_{D_i}$. We have an exact sequence

$$0 \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/S}(\log) \xrightarrow{\oplus_i \mathrm{res}_i} \bigoplus_{i=1}^m O_{D_i} \longrightarrow 0.$$

2. The map $O_X \to \Omega^1_{X/S}(\log)$ defined by $1 \mapsto d\log\pi$ where π is a uniformizer of K is indepenent of the choice of π and induces a map $O_{X_s} \to \Omega^1_{X/S}(\log)$. We have an exact sequence

$$0 \longrightarrow O_{X_s} \longrightarrow \Omega^1_{X/S}(\log) \longrightarrow \Omega^1_{X/S}(\log/\log) \longrightarrow 0.$$

3. The kernel and cokernel of the map $\Omega^1_{X/S} \to \Omega^1_{X/S}(\log/\log)$ are isomorphic respectively to the kernel and cokernel of the map $O_{X_s} \to \bigoplus_{i=1}^m O_{D_i}$ sending 1 to $(l_1, \ldots, l_m)$.

Lemma 1.2.1 is a special case of Lemma 1.19.3 where $X = \mathrm{Spec}\ O_L$.

Proof of Lemma 1.19. 1. The map $\mathrm{res}_i : \Omega^1_{X/S}(\log) \to O_{D_i}$ is well-defined since $a \cdot \mathrm{ord}_i a = 0$ for a local section a of $O_X \cap j_*O^\times_{X_K}$. We show the exact sequence. The question is local on X. Let $U \to P$ and $\Omega^1_{P/S}(\log)$ be as in Lemma 1.17.2 and Corollary 1.18.2. Then we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{U/P} & \longrightarrow & \Omega^1_{P/S}|_U & \longrightarrow & \Omega^1_{U/S} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{U/P} & \longrightarrow & \Omega^1_{P/S}(\log)|_U & \longrightarrow & \Omega^1_{U/S}(\log) & \longrightarrow & 0.
\end{array}$$

The assertion follows from this and the exact sequence

$$0 \longrightarrow \Omega^1_{P/S}|_U \longrightarrow \Omega^1_{P/S}(\log)|_U \xrightarrow{\oplus_i \mathrm{res}_i} \bigoplus_{i=1}^r O_{D_i \cap U} \longrightarrow 0,$$

by the snake lemma.

2. If $\pi' = u\pi$ is another uniformizer, the difference $d\log\pi' - d\log\pi = u^{-1}du$ is 0 in $\Omega^1_{X/S}(\log)$. Hence the map is independent of π. Since $\pi d\log\pi = d\pi = 0$, it factors through O_{X_s}. We show the exact sequence. It suffices to show that the surjection $O_{X_s} \to \mathrm{Ker}(\Omega^1_{X/S}(\log) \to \Omega^1_{X/S}(\log/\log))$ is an isomorphism. Hence, it is reduced to showing that $\mathrm{Ker}(\Omega^1_{X/S}(\log) \to \Omega^1_{X/S}(\log/\log))$ is an invertible O_{X_s}-module. The question is local on X. The assertion follows from the diagram (1) in Corollary 1.18.2 by the snake lemma.

3. The composition of $O_X \to \Omega^1_{X/S}(\log) : 1 \mapsto d\log\pi$ with $\oplus_i \mathrm{res}_i : \Omega^1_{X/S}(\log) \to \bigoplus_{i=1}^m O_{D_i}$ sends 1 to $(l_1, \dots, l_m)$. It follows from this and 1 and 2 above by the snake lemma.

To show the equivalence of Corollary 1.11 and Theorem 1.15, we introduce a notation. Let $D_1, \dots, D_m$ be the irreducible components of $D = (X_F)_{\mathrm{red}}$. For a non-empty subset $I \subset \{1, \dots, m\}$, we put $D_I = \bigcap_{i\in I} D_i$ and $E_I = D_I \cap \bigcup_{i\notin I} D_i$. The scheme D_I is proper and smooth over F of dimension $n - \#I$ and E_I is a divisor of D_I with simple normal crossings. We will prove the following equality which implies the equivalence.

Lemma 1.20 *Let X be a regular flat scheme of dimension n of finite type over O_K. Assume that the reduced closed fiber of X has simple normal crossings and the generic fiber is smooth. Then we have an equality in $CH_0(X_s)$*

$$\begin{aligned}&(c_n{}^X_{X_s}(\Omega^1_{X/S}(\log/\log)) - c_n{}^X_{X_s}(\Omega^1_{X/S})) \cap [X]\\ = &\left(\sum_{r=1}^{n} \sum_{I\subset\{1,\dots,m\},\#I=r} (-1)^r c_{n-r}(\Omega^1_{D_I/F}(\log E_I)) \cap [D_I]\right) + c_{n-1}(\Omega^1_{X/S}(\log/\log)) \cap [X_s].\end{aligned}$$

Proof of equivalence of Theorems 1.10 and 1.15. We prove it admitting Lemma 1.20. By the definition of Artin conductor, it is sufficient to show that $\chi(X_{\bar K}) - \chi(X_{\bar s})$ is $-(-1)^n$-times the degree of the right hand side of the equality in Lemma 1.20. Hence it suffices to show

$$\begin{aligned}\chi(X_{\bar s}) &= \sum_{r=1}^{n} \sum_{I\subset\{1,\dots,m\},\#I=r} \chi((D_I - E_I)_{\bar s})\\ &= \sum_{r=1}^{n} \sum_{I\subset\{1,\dots,m\},\#I=r} \deg(-1)^{n-r} c_{n-r}(\Omega^1_{D_I/F}(\log E_I)) \cap [D_I],\\ \chi(X_{\bar K}) &= \deg(-1)^{n-1} c_{n-1}(\Omega^1_{X/S}(\log/\log)) \cap [X_s].\end{aligned}$$

For the first equality, it is enough to apply Lemma below. The second equality is derived from the Lefschetz trace formula $\chi(X_{\bar K}) = (-1)^{n-1} \deg c_{n-1}(\Omega^1_{X_K/K}) \cap [X_K]$ and the equality $\deg c_{n-1}(\Omega^1_{X_K/K}) \cap [X_K] = \deg c_{n-1}(\Omega^1_{X/S}(\log/\log)) \cap [X_s]$.

Lemma 1.21 *Let V be a proper smooth scheme of dimension n over a perfect field F and D be a divisor of V with simple normal crossings. Then we have*

$$\chi(V_{\bar F} - D_{\bar F}) = \deg(-1)^n c_n(\Omega^1_{V/F}(\log D)).$$

Proof of Lemma 1.21. Let $D_1, \dots, D_r$ be the irreducible components of the divisor D and $\mathrm{res}_i : \Omega^1_{V/F}(\log D) \to O_{D_i}$ be the residue map. Then we have an exact sequence

$$0 \longrightarrow \Omega^1_{V/F} \longrightarrow \Omega^1_{V/F}(\log D) \xrightarrow{\oplus_i \mathrm{res}_i} \bigoplus_{i=1}^r O_{D_i} \longrightarrow 0.$$

Hence we have

$$c(\Omega^1_{V/F})\cap[V]=c(\Omega^1_{V/F}(\log D))\prod_{i=1}^{r}c(O_{D_i})^{-1}\cap[V]=c(\Omega^1_{V/F}(\log D))\prod_{i=1}^{r}(1-[D_i])$$
$$=\sum_{m=0}^{n}\sum_{I\subset\{1,\ldots,r\},\#I=m}(-1)^m c(\Omega^1_{V/F}(\log D))\cap[D_I].$$

By the exact sequence

$$0\longrightarrow\Omega^1_{D_I/F}(\log E_I)\longrightarrow\Omega^1_{V/F}(\log D)|_{D_I}\longrightarrow\bigoplus\nolimits_{i\in I}O_{D_I}\longrightarrow 0,$$

we have $c(\Omega^1_{V/F}(\log D))\cap[D_I]=c(\Omega^1_{D_I/F}(\log E_I))\cap[D_I]$. Hence we have

$$(-1)^n c_n(\Omega^1_{V/F})\cap[V]=\sum_{m=0}^{n}\sum_{I\subset\{1,\ldots,r\},\#I=m}(-1)^{n-m}c_{n-m}(\Omega^1_{D_I/F}(\log E_I))\cap[D_I].$$

On the other hand, we have

$$\chi(V_{\bar F})=\sum_{m=0}^{n}\sum_{I\subset\{1,\ldots,r\},\#I=m}\chi((D_I-E_I)_{\bar F}).$$

By the Lefschetz trace formula $\chi(V_{\bar F})=(-1)^n c_n(\Omega^1_{V/F})\cap[V]$, the left hand sides of the 2 equalities are equal. Hence the assertion follows by induction on $\dim V$.

To complete the proof of the equivalence, we prove Lemma 1.20. The proof is similar to that of Lemma 1.21.

Proof of Lemma 1.20. Let $\mathcal{K}$ and $\mathcal{C}$ denote the kernel and the cokernel of the canonical map $\Omega^1_{X/S}\to\Omega^1_{X/S}(\log/\log)$ respectively. By Proposition 1.3.4 and Lemma 1.19.3, we have an equality

$$c^X_{X_s}(\Omega^1_{X/S}(\log/\log))^{-1}c^X_{X_s}(\Omega^1_{X/S})\cap[X]=c^X_{X_s}(\mathcal{K})^{-1}c^X_{X_s}(\mathcal{C})\cap[X]$$
$$=c^X_{X_s}(O_{X_s})\prod_{i=1}^{r}c^X_{X_s}(O_{D_i})^{-1}\cap[X].$$

By Corollary 1.6 and the equalities $[X_s]^2=[X_s][D_i]=0$, the right hand side is equal to

$$\prod_{i=1}^{r}(1-[D_i])+[X_s]=\sum_{m=0}^{n}\sum_{I\subset\{1,\ldots,r\},\#I=m}(-1)^m[D_I]+[X_s].$$

Subtracting $[X]=[D_\emptyset]$ and multiplying $c^X_{X_F}(\Omega^1_{X/S}(\log/\log))$ to the both sides, we obtain an equality in $CH_*(X_F)$

$$(c^X_{X_F}(\Omega^1_{X/S})-c^X_{X_F}(\Omega^1_{X/S}(\log/\log)))\cap[X]$$
$$=\sum_{m=1}^{n}\sum_{I\subset\{1,\ldots,r\},\#I=m}(-1)^m c(\Omega^1_{X/S}(\log/\log))\cap[D_I]+c(\Omega^1_{X/S}(\log/\log))\cap[X_s].$$

By Lemma 1.19.2 and $[X_s][D_I]=0$, we have an equality $c(\Omega^1_{X/S}(\log/\log))\cap[D_I]=c(\Omega^1_{X/S}(\log))\cap[D_I]$. Hence Lemma 1.20 is reduced to the equality $c(\Omega^1_{X/S}(\log))\cap[D_I]=c(\Omega^1_{D_I/F}(\log E_I))\cap[D_I]$. It follows from Lemma below.

Lemma 1.22 *Let $I\subset\{1,\ldots,m\}$ be a non-empty subset. Then the O_{D_I}-module $\Omega^1_{X/S}(\log)|_{D_I}=\Omega^1_{X/S}(\log)\otimes_{O_X}O_{D_I}$ is locally free and we have an exact sequence*

$$0\longrightarrow\Omega^1_{D_I/F}(\log E_I)\longrightarrow\Omega^1_{X/S}(\log)|_{D_I}\xrightarrow{\oplus_i\mathrm{res}_i}\bigoplus_{i\in I}O_{D_I}\longrightarrow 0.$$

There is an isomorphism $O_{D_I}\to\mathcal{T}or_1^{O_X}(\Omega^1_{X/S}(\log),O_{D_I})$. We have $\mathcal{T}or_q^{O_X}(\Omega^1_{X/S}(\log),O_{D_I})=0$ for $q\neq 0,1$.

Proof of Lemma 1.22. By Corollary 1.18.2, the Tor-sheaf $\mathcal{T}or_q^{O_X}(\Omega^1_{X/S}(\log),O_{D_I})$ vanishes for $q\neq 0,1$. Let $U\to P$ be a regular immersion as above. Then by the commutative diagram (1) in Corollary 1.18.2, we see that the maps $\Omega^1_{P/S}(\log)|_{U\cap D_I}\to\Omega^1_{X/S}(\log)|_{U\cap D_I}$ and $N_{U/P}|_{U\cap D_I}\to\mathcal{T}or_1^{O_X}(\Omega^1_{X/S}(\log),O_{D_I})|_{U\cap D_I}$ are isomorphisms. Hence the O_{D_I}-module $\Omega^1_{X/S}(\log)|_{D_I}$ is locally free and $\mathcal{T}or_1^{O_X}(\Omega^1_{X/S}(\log),O_{D_I})$ is invertible.

We show the exact sequence. We put $U_I=X-\bigcup_{i\notin I}D_i$ and

$$\Omega^1_{X/S}(\log_I)=\left(\Omega^1_{X/S}\oplus(O_X\otimes j_{I*}O^\times_{U_I})\right)/(da-a\otimes a: a\in O_X\cap j_*O^\times_{U_I})$$

where $j_I:U_I\to X$ is the open immersion. Then we have an exact sequence

$$\Omega^1_{X/S}(\log_I)|_{D_I}\longrightarrow\Omega^1_{X/S}(\log)|_{D_I}\xrightarrow{\oplus_i\mathrm{res}_i}\bigoplus_{i\in I}O_{D_I}\longrightarrow 0$$

and a surjection $\Omega^1_{X/S}(\log_I)|_{D_I}\to\Omega^1_{D_I/F}(\log E_I)$. To derive the exact sequence from this, it suffices to show that the map $\Omega^1_{X/S}(\log_I)|_{D_I}\to\Omega^1_{X/S}(\log)|_{D_I}$ is factored by the surjection $\Omega^1_{X/S}(\log_I)|_{D_I}\to\Omega^1_{D_I/F}(\log E_I)$ and that the induced map $\Omega^1_{D_I/F}(\log E_I)\to\Omega^1_{X/S}(\log)|_{D_I}$ is an injection. The question is local on X. Let $U\to P$ be an immersion and $\Delta_{(I)}$ be the divisor of P with relative normal crossings defined by $\prod_{i\in I}T_i$ as in Corollary 1.18.2. We put $\Omega^1_{P/S}(\log_I)=\Omega^1_{P/S}(\log\Delta_{(I)})$. Then we have an exact sequence

$$0\longrightarrow\Omega^1_{P/S}(\log_I)\longrightarrow\Omega^1_{P/S}(\log)\xrightarrow{\oplus_i\mathrm{res}_i}\bigoplus_{i\in I}O_{\Delta_i}\longrightarrow 0$$

where Δ_i is the divisor $T_i=0$. By restricting it to $U\cap D_I$, we obtain an exact sequence

$$0\longrightarrow\Omega^1_{D_I}(\log E_I)|_{U\cap D_I}\longrightarrow\Omega^1_{P/S}(\log)|_{U\cap D_I}\xrightarrow{\oplus_i\mathrm{res}_i}\bigoplus_{i\in I}O_{U\cap D_I}\longrightarrow 0.$$

By the isomorphism $\Omega^1_{P/S}(\log)|_{U\cap D_I}\to\Omega^1_{X/S}(\log)|_{U\cap D_I}$ and the surjection $\Omega^1_{P/S}(\log_I)|_{U\cap D_I}\to\Omega^1_{X/S}(\log_I)|_{U\cap D_I}$, it implies the assertion.

We define an isomorphism $O_{D_I}\to\mathcal{T}or_1^{O_X}(\Omega^1_{X/S}(\log),O_{D_I})$. By the exact sequence $0\to O_{X_s}\to\Omega^1_{X/S}(\log)\to\Omega^1_{X/S}(\log/\log)\to 0$ and by the vanishing of $\mathcal{T}or_2^{O_X}(\Omega^1_{X/S}(\log/\log),O_{D_I})$, we get an exact sequence

$$0\longrightarrow\mathcal{T}or_1^{O_X}(O_{X_s},O_{D_I})\longrightarrow\mathcal{T}or_1^{O_X}(\Omega^1_{X/S}(\log),O_{D_I})\longrightarrow\mathcal{T}or_1^{O_X}(\Omega^1_{X/S}(\log/\log),O_{D_I}).$$

The first two O_{D_I}-modules are invertible. The last one is locally a submodule of an invertible O_{D_I}-module and is torsion free. Hence the cokernel of the injection $O_{D_I} \simeq \mathcal{T}or_1^{O_X}(O_{X_s}, O_{D_I}) \to \mathcal{T}or_1^{O_X}(\Omega^1_{X/S}(\log), O_{D_I})$ is 0 and the map is an isomorphism.

1.5. Review on cotangent complexes of schemes locally of complete intersection.

We recall some generality on cotangent complex of a scheme locally of complete intersection and prove Proposition 1.16 and Lemma 1.25. Basic reference is [14] Chapitre II.

Definition 1.23 *We say a scheme X over S is locally of complete intersection if, for each $x \in X$, there exist an open neighborhood U of x in X and a regular immersion $U \to P$ into a smooth scheme P over S.*

If a scheme locally of finite type over a regular noetherian scheme S is regular, it is locally of complete intersection over S. If X is locally of complete intersection over S and if $i : X \to P$ is an immersion over S into a smooth scheme P over S, then the immersion i is regular.

We briefly recall the definition and basic properties of the cotangent complex $L_{X/S}$ of a scheme $f : X \to S$ locally of complete intersection over S. First we give a local description of the cotangent complex. Let $i : X \to P$ be a regular immersion into a smooth scheme P over S and let $N_{X/P}$ denote the conormal sheaf. The cotangent complex $L_{X/S,P}$ of X over S with respect to the immersion $X \to P$ is defined by $L_{X/S,P} = [N_{X/P} \to \Omega^1_{P/S}|_X]$ where $\Omega^1_{P/S}|_X$ is put on degree 0. The cotangent complex $L_{X/S}$ is defined to be a complex which is locally quasi-isomorphic to $L_{X/S,P}$ as follows.

Let A be a sheaf of O_S-algebras on X and $A \to O_X$ be a surjective homomorphism of O_S-algebras. Let $N_{X/A}$ denote the O_X-module $\mathrm{Ker}(A \to O_X) \otimes_A O_X = \mathrm{Ker}(A \to O_X)/(\mathrm{Ker}(A \to O_X))^2$. The derivation defines an O_X-linear map $d : N_{X/A} \to \Omega^1_{A/O_S} \otimes_A O_X$. We define a complex $L_{X,A}$ of O_X-modules to be $[N_{X/A} \to \Omega^1_{A/O_S} \otimes_A O_X]$ putting $\Omega^1_{A/O_S} \otimes_A O_X$ on degree 0. Let $A_{X/S} = f^{-1}O_S[O_X]$ be the sheaf of O_S-algebras on X associated to the presheaf $U \mapsto \Gamma(U, f^{-1}O_S)[\Gamma(U, O_X)]$ where $\Gamma(U, f^{-1}O_S)[\Gamma(U, O_X)]$ denotes the free $\Gamma(U, f^{-1}O_S)$-algebra generated by $\Gamma(U, O_X)$. For a section $\varphi \in \Gamma(U, O_X)$, let (φ) denote the corresponding section of $\Gamma(U, A_{X/S})$. We have a canonical surjection $A_{X/S} \to O_X$ of $f^{-1}O_S$-algebras sending (φ) to φ. We define the cotangent complex $L_{X/S}$ to be $L_{X,A_{X/S}}$. For a sheaf A of algebras and a sheaf $\mathcal{F}$ of sets on X, let $A^{(\mathcal{F})}$ be the sheaf of A-modules associated to the presheaf $U \mapsto \Gamma(U, A)^{(\Gamma(U,\mathcal{F}))}$ where $\Gamma(U, A)^{(\Gamma(U,\mathcal{F}))}$ is the free $\Gamma(U, A)$-module generated by $\Gamma(U, \mathcal{F})$. The sheaf $\Omega^1_{A_{X/S}/O_S}$ is canonically identified with the $A_{X/S}$-module $A_{X/S}^{(O_X)}$. Hence, the cotangent complex $L_{X/S}$ is identified as $[N_{X/A_{X/S}} \xrightarrow{d} O_X^{(O_X)}]$. The complexes $L_{X/S}$ and $L_{X/S,P}$ are related as follows. Let $i : X \to P$ be a regular immersion into a smooth scheme P over S. Then the complex $L_{X/S,P}$ is identified with $L_{X,i^{-1}O_P}$. The canonical maps $L_{X/S,P} = L_{X,i^{-1}O_P} \leftarrow L_{X,f^{-1}O_S[i^{-1}O_P]} \to L_{X,A_{X/S}} = L_{X/S}$ are quasi-isomorphisms.

The quasi-isomorphism $L_{X/S} \to L_{X/S,P}$ implies that the cohomology sheaf $\mathcal{H}_0(L_{X/S})$ is canonically isomorphic to $\Omega^1_{X/S}$. Assume there exists a dense open subscheme $U \subset X$ smooth over S. Either if $X = U$ or if X is reduced, the cohomology sheaf $\mathcal{H}_1(L_{X/S})$ is 0 and the canonical map $L_{X/S} \to \Omega^1_{X/S}$ is a quasi-isomorphism.

The cotangent complexes have the following functoriality. Let Y be a scheme locally of complete intersection over a scheme T and $g : X \to Y$ be a morphism compatible with $S \to T$. Then the canonical map $g^{-1}A_{Y/T} \to A_{X/S}$ induces a map $g^*L_{Y/T} \to L_{X/S}$. Let

$$\begin{array}{ccccc} X & \xrightarrow{i} & P & \longrightarrow & S \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow{j} & Q & \longrightarrow & T \end{array}$$

be a commutative diagram of schemes where $i : X \to P$ and $j : Y \to Q$ are regular immersions and $P \to S$ and $Q \to T$ are smooth. Then the canonical maps $f^*N_{Y/Q} \to N_{X/P}$ and $f^*(\Omega^1_{Q/T}|_Y) \to \Omega^1_{P/S}|_X$ define a morphism $Lf^*L_{Y/T,Q} \to L_{X/S,P}$ of complex and the diagram

$$\begin{array}{ccccc} Lf^*L_{Y/T,Q} & \longleftarrow & Lf^*L_{Y,O_T[j^{-1}O_Q]} & \longrightarrow & Lf^*L_{Y/T} \\ \downarrow & & \downarrow & & \downarrow \\ L_{X/S,P} & \longleftarrow & L_{X,O_S[i^{-1}O_P]} & \longrightarrow & L_{X/S} \end{array}$$

is commutative. The horizontal arrows are quasi-isomorphisms.

Lemma 1.24 ([14] Chapitre II Proposition 2.1.2) *Let X and Y be schemes locally of complete intersection over S and $f : X \to Y$ be a morphism over S locally of complete intersection. Then we have a distinguished triangle*

$$Lf^*L_{Y/S} \longrightarrow L_{X/S} \longrightarrow L_{X/Y} \longrightarrow .$$

Lemma 1.25 *Let K be a discrete valuation field with perfect residue field F and X be a regular flat scheme over $S = \text{Spec } O_K$ with smooth generic fiber. Let $i : Z \to X$ be the closed immersion defined by the ideal $\text{Ann } \Omega^n_{X/S}$ and $\mathcal{L}_Z$ be the invertible O_Z-module $L^1 i^*\Omega^1_{X/S}$. Let W be a normal scheme of finite type over $s = \text{Spec } F$ and $\varphi : W \to Z$ be a morphism over S. Then,*

1. There exists a canonical isomorphism $\varphi^*\mathcal{L}_Z = L^1(i \circ \varphi)^*\Omega^1_{X/S} \to N_{s/S} \otimes O_W \simeq O_W$ of invertible O_W-modules.

2. The chern class $c_1(\mathcal{L}_Z) \in CH^1(Z \to Z)$ is 0.

Proof. 1. The O_W-module $\varphi^*\mathcal{L}_Z = L^1(i \circ \varphi)^*\Omega^1_{X/S}$ is invertible by Corollary 1.18.1. Therefore, to define an isomorphism $L^1(i \circ \varphi)^*\Omega^1_{X/S} \to N_{s/S} \otimes O_W$ of invertible O_W-modules, we may shrink W to an open subset containing all the points of codimension 1. Shrinking W, we may assume W is smooth over s. Applying Lemma 1.24 to $W \to X \to S$, we obtain a distinguished triangle

$$L(i \circ \varphi)^*L_{X/S} \longrightarrow L_{W/S} \longrightarrow L_{W/X} \longrightarrow . \qquad (2)$$

Here we have $L_{X/S} = \Omega^1_{X/S}$. We compute $L_{W/S}$ by applying Lemma 1.24 to $W \to s \to S$. We obtain a distinguished triangle $L_{s/S} \otimes O_W \to L_{W/S} \to L_{W/s} \to$. Since $L_{s/S} = N_{s/S}[1]$ and

$L_{W/s} = \Omega^1_{W/s}$, we have $\mathcal{H}_0(L_{W/S}) = \Omega^1_{W/s}$ and $\mathcal{H}_1(L_{W/S}) = N_{s/S} \otimes_F O_W$. Taking the cohomology $\mathcal{H}_1$ of the distinguished triangle (2) above, we obtain an exact sequence

$$0 \longrightarrow L^1(i \circ \varphi)^*\Omega^1_{X/S} \xrightarrow{a} N_{s/S} \otimes_F O_W \xrightarrow{b} \mathcal{H}_1(L_{W/X}).$$

We show that the map a is an isomorphism. By locally embedding W in a smooth scheme over X, we see that the O_W-module $\mathcal{H}_1(L_{W/X})$ is a subsheaf of a locally free sheaf and hence is torsion free. On the other hand, since a is injective, the cokernel of a is of torsion. Hence the map b is 0 and a is an isomorphism.

2. For a scheme Z' of finite type over Z, the Chow group $CH_i(Z')$ is generated by $\pi_*[W]$ where $\pi : W \to Z'$ runs the normalization of an integral closed subscheme of Z' of dimension i. By 1, we have $c_1(\mathcal{L}_Z) \cap \pi_*[W] = \pi_*(c_1(\pi^*\mathcal{L}_Z) \cap [W]) = \pi_*[W]$ and the assertion follows.

Proof of Proposition 1.16. 1. By Corollary 1.18.1, the O_X-module Ω^1_{X/O_K} is locally generated by n sections. By Lemma 1.4, there exists a locally free resolution $0 \to \mathcal{L} \to \mathcal{E} \to \Omega^1_{X/O_K} \to 0$. We have rank $\mathcal{E}$ − rank $\mathcal{L} = n - 1$. We apply Corollary 1.8 to the blow-up $X' \to X$ and to the complex $\mathcal{K} = [\mathcal{L} \to \mathcal{E}]$ quasi-isomorphic to Ω^1_{X/O_K}. Let $i : Z \to X$ and $\pi_D : D \to Z$ be the canonical maps and let $\mathcal{L}_Z = L^1 i^*\Omega^1_{X/S}$. Then, by applying Corollary 1.8, we obtain $c_n{}^X_Z(\Omega^1_{X/S}) \cap [X] = \pi_*(c_{n-1}(\mathcal{E}'|_D \otimes (\pi_D^*\mathcal{L}_Z)^{\otimes -1}) \cap [D])$. By Lemma 1.25.2, the right hand side is further equal to $\pi_*(c_{n-1}(\mathcal{E}'|_D) \cap [D])$

2. Let $i : Z \to X$ and $\pi_D : D \to Z$ be the canonical maps and put $\mathcal{L}_Z = L^1 i^*\Omega^1_{X/S}(\log / \log)$. Similarly as above, we obtain $c_n{}^X_Z(\Omega^1_{X/S}(\log / \log)) \cap [X] = \pi_*(c_{n-1}(\mathcal{E}'|_D \otimes (\pi_D^*\mathcal{L}_Z)^{\otimes -1}) \cap [D])$. To complete the proof, it suffices to show the following Lemma.

Lemma 1.26 *Let X be a regular flat and separated scheme of dimension n of finite type over* Spec O_K *with smooth generic fiber. Assume the reduced closed fiber $X_{s,\mathrm{red}}$ has simple normal crossings. Let $i : Z \to X$ be the closed immersion defined by the ideal* Ann $\Lambda^n\Omega^1_{X/S}(\log / \log)$ *and let $\mathcal{L}_Z$ be the invertible O_Z-module $L^1 i^*\Omega^1_{X/S}(\log / \log)$. Let $\bar{Z} = Z_{\mathrm{red}}$ and $\bar{i} : \bar{Z} \to X$ be the immersion.*

1. *On $\bar{Z}$, there is a canonical isomorphism $\mathcal{L}_Z|_{\bar{Z}} = L^1\bar{i}^*\Omega^1_{X/S}(\log / \log) \to O_{\bar{Z}}$.*
2. *The chern class $c_1(\mathcal{L}_Z) \in CH^1(Z \to Z)$ is 0.*

Proof of Lemma 1.26. 1. Applying $L\bar{i}^*$ to the exact sequence in Lemma 1.19.2, we obtain a long exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & O_{\bar{Z}} & \longrightarrow & L^1\bar{i}^*\Omega^1_{X/S}(\log) & \longrightarrow & L^1\bar{i}^*\Omega^1_{X/S}(\log / \log) & \longrightarrow & \\ & & O_{\bar{Z}} & \longrightarrow & \Omega^1_{X/S}(\log)|_{\bar{Z}} & \longrightarrow & \Omega^1_{X/S}(\log / \log)|_{\bar{Z}} & \longrightarrow & 0. \end{array}$$

It follows from the commutative diagram (1) in Corollary 1.18.2 that the map $\Omega^1_{X/S}(\log)|_{\bar{Z}} \to \Omega^1_{X/S}(\log / \log)|_{\bar{Z}}$ is an isomorphism and the map $L^1\bar{i}^*\Omega^1_{X/S}(\log) \to L^1\bar{i}^*\Omega^1_{X/S}(\log / \log)$ is the 0-map. Hence the boundary map $L^1\bar{i}^*\Omega^1_{X/S}(\log / \log) \to O_{\bar{Z}}$ is an isomorphism.

2. It follows immediately from 1.

1.6. Blow-up

To complete the proof of the equivalence of Theorem 1.10 and Corollary 1.11, we show Propositions 1.12 and 1.13.
Proof of Proposition 1.12. More precisely, we prove the following equality.

Proposition 1.27 *Let the notation be as in Proposition 1.12. Let n be the dimension of X, c be the codimension of C in X and $i_s : C \to X_s$ be the immersion. Then, we have an equality in $CH_0(X_s)$*

$$\pi_*(c_n{}^{X'}_{X'_s}(\Omega^1_{X'/S}) \cap [X']) - c_n{}^{X}_{X_s}(\Omega^1_{X/S}) \cap [X] = (-1)^c(c-1)i_{s*}(c_{n-c}(\Omega^1_{C/F}) \cap [C]).$$

We deduce Proposition 1.12 from 1.27. Let $E = X' \times_X C$ be the exceptional divisor. By the definition of the Artin conductor, we have

$$-(\mathrm{Art}(X'/S) - \mathrm{Art}(X/S)) = \chi(X'_{\bar s}) - \chi(X_{\bar s}) = \chi(E_{\bar s}) - \chi(C_{\bar s}).$$

Since E is a $\mathbf{P}^{c-1}$-bundle over C, we have $\chi(E_{\bar s}) = c\chi(C_{\bar s})$. By the Lefschetz trace formula $\chi(C_{\bar s}) = \deg(-1)^{n-c}c_{n-c}(\Omega^1_{C/F}) \cap [C]$, it is reduced to Proposition 1.27.
Proof of Proposition 1.27. Let E be the exceptional divisor, $i : C \to X$ and $j : E \to X'$ be the immersion and $\pi_E : E \to C$ be the projection. We have an exact sequence

$$0 \longrightarrow \pi^*\Omega^1_{X/S} \longrightarrow \Omega^1_{X'/S} \longrightarrow j_*\Omega^1_{E/C} \longrightarrow 0$$

and hence an equality $c^{X'}_{X'_s}(\Omega^1_{X'/S}) = c^{X'}_{X'_s}(\pi^*\Omega^1_{X/S})c^{X'}_{X'_s}(j_*\Omega^1_{E/C})$. By Proposition 1.3.5 and the quasi-isomorphism $L\pi^*\Omega^1_{X/S} \to \pi^*\Omega^1_{X/S}$, we have

$$\pi_*(c^{X'}_{X'_s}(\Omega^1_{X'/S}) \cap [X']) - c^{X}_{X_s}(\Omega^1_{X/S}) \cap [X] = i_{s*}(c(Li^*\Omega^1_{X/S}) \cap (\pi_{E*}(c^{X'}_E(j_*\Omega^1_{E/C}) - 1) \cap [X'])). \quad (3)$$

We compute $\pi_{E*}(c^{X'}_E(j_*\Omega^1_{E/C}) - 1) \cap [X'])$ in the right hand side.

Lemma 1.28 *We have an equality in $CH_*(C)$*

$$\pi_{E*}(c^{X'}_E(j_*\Omega^1_{E/C}) - 1) \cap [X']) = (-1)^c(c-1) \cdot c(N_{C/X})^{-1} \cap [C].$$

Proof of Lemma 1.28. Since E is a $\mathbf{P}^{c-1}$-bundle $\mathbf{P}_C(N_{C/X})$, we have an exact sequence $0 \to \Omega^1_{E/C} \to \pi_E^*N_{C/X}(-1) \to O_E \to 0$ and hence an equality $c^{X'}_E(j_*\Omega^1_{E/C}) = c^{X'}_E(j_*\pi_E^*N_{C/X}(-1)) \cdot c^{X'}_E(j_*O_E)^{-1}$. Thus we have

$$\begin{aligned}&(c^{X'}_E(j_*\Omega^1_{E/C}) - 1) \cap [X'] \\ =\ &(c^{X'}_E(j_*\pi_E^*N_{C/X}(-1)) - 1) \cap [X'] + c^{X'}_E(j_*\pi_E^*N_{C/X}(-1))(c^{X}_E(j_*O_E)^{-1} - 1) \cap [X'].\end{aligned}$$

We put $\mathcal{E} = \pi_E^*N_{C/X}(-1)$. By Lemma 1.5, the two terms in the right hand side are equal to $A = \sum_{j=1}^c a_j(\mathcal{E})E^{j-1}c(\mathcal{E}(-E))^{-1} \cap [E]$ and $-B = -c^{X'}_E(\mathcal{E}) \cap [E]$ respectively, where $a_j(\mathcal{E}) = \sum_{k=j}^n \binom{k}{j} c_{n-k}(\mathcal{E}(-E))$. We show

$$\pi_{E*}A = (-1)^{c-1}c(N_{C/X})^{-1} \cap [C] \quad \text{and} \quad \pi_{E*}B = (-1)^{c-1}c \cdot c(N_{C/X})^{-1} \cap [C]$$

to complete the proof. We have $a_j(\mathcal{E}) = \pi_E^* a_j^0$ where $a_j^0 = \sum_{k=j}^c \binom{k}{j} c_{c-k}(N_{C/X})$ since $O(E) = O(-1)$. Hence we have an equality $A = c(\pi_E^* N_{C/X})^{-1} \sum_{j=1}^c \pi_E^* a_j^0 E^{j-1} \cap [E]$ and consequently $\pi_{E*}A = c(N_{C/X})^{-1} \sum_{j=1}^c a_j^0 \pi_{E*}(E^{j-1} \cap [E])$. Since $\pi_{E*}(E^{j-1} \cap [E]) = (-1)^{c-1}[C]$ if $j = c$ and is 0 for $j < c$ and $a_c^0 = 1$, we obtain $\pi_{E*}A = (-1)^{c-1} c(N_{C/X})^{-1} \cap [C]$

We compute $\pi_{E*}B$. We have $B = c_E^{X'}(j_*\mathcal{E}) \cap [E] = c_E(Lj^*j_*\mathcal{E}) \cap [E] = c_E(\mathcal{E}) c_E(\pi_E^* N_{C/X})^{-1} \cap [E]$ since $L^0 j^* j_* \mathcal{E} = \mathcal{E}$, $L^1 j^* j_* \mathcal{E} = \mathcal{E} \otimes O_E(-E) = \pi_E^* N_{C/X}$ and $L^q j^* j_* \mathcal{E} = 0$ for $q \neq 0, 1$. We have $c_E(\mathcal{E}) = c_E(\pi_E^* N_{C/X}(E)) = \sum_{i=0}^c \sum_{j=0}^{c-i} \binom{c-j}{i} \pi_E^* c_j(N_{C/X}) E^i$ and $E^c = -\sum_{j=1}^c \pi_E^* c_j(N_{C/X}) E^{c-j}$. Hence we obtain

$$\begin{aligned} B &= \pi_E^* c(N_{C/X})^{-1} \Big(\sum_{i=0}^{c-1} \sum_{j=0}^{c-i} \binom{c-j}{i} \pi_E^* c_j(N_{C/X}) E^i - \sum_{j=1}^{c} \pi_E^* c_j(N_{C/X}) E^{c-j} \Big) \cap [E] \\ &= \pi_E^* c(N_{C/X})^{-1} \sum_{i=0}^{c-1} \sum_{j=0}^{c-1-i} \binom{c-j}{i} \pi_E^* c_j(N_{C/X}) (E^i \cap [E]). \end{aligned}$$

Similarly as above, we have $\pi_{E*}B = (-1)^{c-1} c \cdot c(N_{C/X})^{-1} \cap [C]$. Thus Lemma 1.28 is proved.

To complete the proof of Proposition 1.27, we show $c(Li^*\Omega^1_{X/S}) = c(\Omega^1_C) c(N_{C/X})$ in $CH^*(C \to C)$. Applying Lemma 1.24 to $C \to X \to S$, we obtain a distinguished triangle $Li^* L_{X/S} \to L_{C/S} \to L_{C/X} \to$. Here we have $L_{X/S} = \Omega^1_{X/S}$ and $L_{C/X} = N_{C/X}[1]$. We compute $L_{C/S}$ by applying Lemma 1.24 to $C \to s \to S$. We obtain a distinguished triangle $L_{s/S} \otimes O_C \to L_{C/S} \to L_{C/s} \to$. Since $L_{s/S} = N_{s/S}[1]$ and $L_{C/s} = \Omega^1_{C/s}$, we have $\mathcal{H}_0(L_{C/S}) = \Omega^1_{C/s}$ and $\mathcal{H}_1(L_{C/S}) \simeq O_C$. Thus we have $c(Li^*\Omega^1_{X/S}) = c(\Omega^1_C) c(N_{C/X})$. Multiplying this to the equality of Lemma 1.28 and combining with the equality (3), we obtain the required equality.

Proof of Proposition 1.13. 1. Since the assertion is etale local, we may assume that the divisor $D = \bigcup_{i=1}^r D_i$ has simple normal crossing and that each irreducible component D_i is defined by one element t_i. The blow-up $X_i \to X$ is described in terms of a partial barycentric subdivision of a simplex as follows (cf §3.1). We regard $\Delta = \{1, \ldots, r\}$ as the set of vertices of a simplex $|\Delta|$. We define a subdivision of $|\Delta|$ as follows. We regard a subset $\tau \subset \Delta$ as the face spanned by τ and further identify it with the barycenter of the face. For each $0 \leq i < n$, we put $\Delta_i = \Delta \amalg \{\tau \subset \Delta | \#\tau > n - i\}$. It is the set of vertices of $|\Delta|$ together with the barycenters of faces with dimension $\geq n - i$. We say a subset $\sigma \subset \Delta_i$ is a face of Δ_i if the following condition is satisfied: The set $\sigma_1 = \sigma \cap \Delta$ consists of at most $n - i$ elements and $(\sigma - \sigma_1) \amalg \{\sigma_1\}$ is totally ordered with respect to the inclusions. The faces of Δ_i defines a subdivision of $|\Delta|$.

Using the faces of Δ_i, the scheme X_i is described as follows. Let $M = \mathbf{N}^r$ and $e_1, \ldots, e_r$ be the standard basis of M. Define a ring homomorphism $\mathbf{Z}[M] \to \Gamma(X, O_X)$ by sending $e_i \mapsto t_i$. Let $e_1^*, \ldots, e_r^*$ be the dual basis of $N = Hom_{\text{monoid}}(M, \mathbf{N})$ and for $\tau \subset \Delta$, we put $e_\tau^* = \sum_{j \in \tau} e_j^* \in N$. Let M^{gp} be the group $\mathbf{Z}^r$ containing M as a submonoid and we identify $N \subset N^{\text{gp}} = Hom(M^{\text{gp}}, \mathbf{Z})$. For a face $\sigma \subset \Delta_i$, we put $M_\sigma = \{x \in M^{\text{gp}} | e_\tau^*(x) \geq 0 \text{ for all } \tau \in \sigma\}$ and $U_\sigma = X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M_\sigma]$. Patching them by the natural inclusions $U_\sigma \to U_{\sigma'}$ for $\sigma \subset \sigma'$, we obtain X_i.

That X_i is regular and D_i has normal crossing follows immediately from the local description above and that the set $\{e_\tau^* | \tau \in \sigma\}$ is a linearly independent subset of M^{gp} for a face $\sigma \subset \Delta_i$. We show V_i' is regular. For a face $\tau \subset \Delta$, let I_τ be the ideal of $\mathbf{Z}[M]$ generated by $\{x \in M | e_j^*(x) >$

0 for some $j \in \tau\}$ and V_τ be the closed subscheme $X \otimes_{\mathbf{Z}[M]} \mathbf{Z}[M]/I_\tau$. We have $V_i = \bigcup_{\#\tau=n-i} V_\tau$. For a face $\tau \subset \Delta$ with $\#\tau = n-i$, let V'_τ be the proper transform of V_τ in X_i. We show $V'_i = \coprod_{\#\tau=n-i} V'_\tau$ and is regular. It is sufficient to show the following. Let $\tau \subset \Delta$ be a face with $n-i$ elements and let $\sigma \subset \Delta_i$ be a face. Then $V'_\tau \cap U_\sigma$ is regular and, if $\tau' \neq \tau$ is another face with $n-i$ elements, then $V'_\tau \cap V'_{\tau'} \cap U_\sigma = \emptyset$. If σ contains $[\tau] = \{\{j\} | j \in \tau\}$ as a subset, let $I_{\tau,\sigma}$ be the ideal of $\mathbf{Z}[M_\sigma]$ generated by $\{x \in M_\sigma | e^*_j(x) > 0$ for some $j \in \tau\}$. The intersection $V'_\tau \cap U_\sigma$ is the closed subscheme $U_\sigma \otimes_{\mathbf{Z}[M_\sigma]} (\mathbf{Z}[M_\sigma]/I_{\tau,\sigma})$ of U_σ if $[\tau] \subset \sigma$ and is empty if otherwise. Thus assertion is proved.

2. By 1, V'_{n-1} is a regular divisor. Since the exceptional divisors are also regular, every irreducible components of the divisor D_{n-1} is regular. Therefore D_{n-1} has simple normal crossings.

2. K**-theoretic localized intersection product.**

In this section, we define K-theoretic localized intersection product, which plays an essential role in the proof of conductor formula. We also establish its basic properties; relation with usual product, Proposition 2.14, excess intersection formula, Proposition 2.16, associativity, Proposition 2.23, and projection formula, Proposition 2.19. As a preliminary, we recall generalities on usual K-theoretic intersection product in §2.1. We state the main results in §2.2 and prove them in §2.3 except the excess intersection formula, Proposition 2.16. A relation between the localized self-intersection class $(\Delta_X, \Delta_X)_S$ introduced in §1.3 and the localized intersection product $[[X, X]]_{X\times_S X}$ defined in Example 2 after Definition 2.13 is given in Corollary 2.18 of Proposition 2.16. The proof of Proposition 2.16, given in §2.5, requires some preliminary in §2.4 on derived exterior power complexes.

2.1. Review on K-theory and intersection product.

We recall generalities on K-theoretic intersection theory. Basic references are [12] and [9].

For a scheme X, let $K(X)$ be the Grothendieck group of the category of locally free O_X-modules of finite rank. It is the quotient of the free abelian group generated by the isomorphism classes $[\mathcal{E}]$ of locally free O_X-modules of finite rank by the relations $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$ for exact sequences $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$. For a noetherian scheme X, let $G(X)$ be the Grothendieck group of the category of coherent O_X-modules. It is the quotient of the free abelian group generated by the isomorphism classes $[\mathcal{F}]$ of coherent O_X-modules by the relations $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$ for exact sequences $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. We say a chain complex $\mathcal{G} = (\mathcal{G}_q, d_q)_q$ of O_X-modules is coherent if the homology sheaves $\mathcal{H}_q(\mathcal{G})$ are 0 except for finitely many q and are coherent for all q. For a coherent complex $\mathcal{G}$, its class $[\mathcal{G}] \in G(X)$ is defined as the alternating sum $\sum_q (-1)^q [\mathcal{H}_q(\mathcal{G})]$. For a distinguished triangle $\to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to$ of coherent complexes, we have $[\mathcal{G}] = [\mathcal{G}'] + [\mathcal{G}'']$.

We have a canonical map $K(X) \to G(X)$ defined by $[\mathcal{E}] \mapsto [\mathcal{E}]$ for a locally free O_X-module $\mathcal{E}$. If X is regular, noetherian and separated, then the canonical map $K(X) \to G(X)$ is an isomorphism. The inverse is given as follows. Let $\mathcal{F}$ be a coherent O_X-module and $(\mathcal{E}_\bullet)$ be a resolution as in Lemma 1.4. Then the inverse map sends $[\mathcal{F}]$ to $\sum_{q=0}^n (-1)^q [\mathcal{E}_q]$.

The multiplication on $K(X)$ is defined by the tensor product $[\mathcal{E}] \cdot [\mathcal{E}'] = [\mathcal{E} \otimes_{O_X} \mathcal{E}']$. If X is noetherian, $G(X)$ is a $K(X)$-module by the multiplication $[\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E} \otimes_{O_X} \mathcal{F}]$. Assume X is regular, noetherian and separated and identify $G(X) = K(X)$. Then the multiplication on $G(X)$ is given by $[\mathcal{F}] \cdot [\mathcal{F}'] = \sum_{q=0}^n (-1)^q [\mathcal{T}or^q_{O_X}(\mathcal{F}, \mathcal{F}')]$ where $n = \dim X$.

Let $V \to X, W \to X$ be morphisms of schemes. Let $W_V = V \times_X W$ be the fiber product and $p_1 : W_V \to V, p_2 : W_V \to W$ be the projections. For a complex $\mathcal{F}$ of O_V-modules and $\mathcal{G}$ of O_W-modules satisfying $\mathcal{H}_q(\mathcal{F}) = 0$ and $\mathcal{H}_q(\mathcal{G}) = 0$ for sufficiently small q, let $\mathcal{F} \otimes^L_{O_X} \mathcal{G}$ denote the complex $Lp_1^*\mathcal{F} \otimes^L_{O_{W_V}} Lp_2^*\mathcal{G}$ and put $\mathcal{T}or_q^{O_X}(\mathcal{F}, \mathcal{G}) = \mathcal{H}_q(\mathcal{F} \otimes^L_{O_X} \mathcal{G})$. Locally, it is computed as follows. If $X = \text{Spec } A, V = \text{Spec } B, W = \text{Spec } C$ are affine and if $\mathcal{F} = M^\sim, \mathcal{G} = N^\sim$ are quasi-coherent sheaves associated to a B-module M and to a C-module N respectively, then $\mathcal{T}or_q^{O_X}(\mathcal{F}, \mathcal{G})$ is nothing but the quasi-coherent sheaf associated to the $B \otimes_A C$-module $Tor_q^A(M, N)$. If X, V, W are noetherian and if X is regular of dimension n, a bilinear map $(\ ,\ )_X : G(V) \times G(W) \to G(W_V)$ is defined by $([\mathcal{F}], [\mathcal{G}])_X = [\mathcal{F} \otimes^L_{O_X} \mathcal{G}] = \sum_{q=0}^n (-1)^q [\mathcal{T}or_q^{O_X}(\mathcal{F}, \mathcal{G})]$. We write $([O_V], [O_W])_X = (V, W)_X$ for short.

The γ-filtration $F^n K(X)$ on $K(X)$ is defined as follows. There is a canonical map $\lambda_t : K(X) \to$

$1+tK(X)[[t]] \subset K(X)[[t]]^\times$ sending the class $[\mathcal{E}]$ of a locally free O_X-module $\mathcal{E}$ to $\sum_q [\Lambda^q \mathcal{E}] t^q$. For $x \in K(X)$, we put $\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x) = 1 + \sum_{n>0} \gamma^n(x) t^n$. For a locally free O_X-module $\mathcal{E}$ of rank n, we have

$$\gamma_t([\mathcal{E}]-n) = \sum_{q=0}^{n} [\Lambda^q \mathcal{E}] t^q (1-t)^{n-q} = \sum_{r=0}^{n} (\sum_{q=0}^{r} (-1)^{r-q} \binom{n-q}{r-q} [\Lambda^q \mathcal{E}]) t^r. \tag{4}$$

In particular, if $\mathcal{L}$ is invertible, we have $\gamma_t([\mathcal{L}]-1) = 1 + ([\mathcal{L}]-1)t$. For $n=1$, $F^1K(X)$ is defined to be the kernel of the map $K(X) \to \mathbf{Z}^{\pi_0(X)}$ sending $\mathcal{E}$ to rank $\mathcal{E}$. For $n \geq 1$, $F^nK(X)$ is defined as the subgroup generated by the elements of the form $\gamma^{n_1}(x_1) \cdots \gamma^{n_r}(x_r)$ where $x_i \in F^1K(X)$ and $\sum_i n_i \geq n$. We put $F^0K(X) = K(X)$. We have $F^nK(X) \cdot F^mK(X) \subset F^{m+n}K(X)$.

Assume X is of finite type over a regular noetherian base scheme S of finite dimension. The topological filtration on $G(X)$ is defined as follows. It is called the lower filtration in [9] Chapter VI §5. For an integer $n \geq 0$, let $F_nG(X)$ be the subgroup of $G(X)$ generated by the classes $[\mathcal{F}]$ of coherent O_X-modules $\mathcal{F}$ such that the dimension of the support of $\mathcal{F}$ is at most n. The γ-filtration and the topological filtration are related as follows.

Lemma 2.1 *Let X be a scheme of finite type over a regular noetherian base scheme S of finite dimension.*

1. ([9] Chapter V Theorem 3.9, Chapter VI Proposition 5.2) *We have $F^nK(X) \cdot F_mG(X) \subset F_{m-n}G(X)$. In particular, if X is of dimension d, the canonical map $K(X) \to G(X)$ sends $F^nK(X)$ into $F_{d-n}G(X)$.*

2. ([9] Chapter VI Proposition 5.5) *If X is regular of pure dimension d and if there exists an ample invertible O_X-module, the induced map $Gr^n_F K(X)_{\mathbf{Q}} \to Gr^F_{d-n} G(X)_{\mathbf{Q}}$ is an isomorphism.*

Let $f : X \to Y$ be a morphism of schemes. The pull-back of locally free sheaves defines a ring homomorphism $f^* : K(Y) \to K(X)$. We have $f^*F^nK(Y) \subset F^nK(X)$. Assume X and Y are noetherian. If f is proper, there is a map $f_* : G(X) \to G(Y)$ sending the class of a coherent O_X-module $\mathcal{F}$ to the class of the coherent complex $Rf_*\mathcal{F}$. Assume f is of finite tor-dimension. Flat maps and regular immersions are examples of maps of finite tor-dimension. For a subscheme Z of Y, we put $Z' = Z \times_Y X$. Then we have a pull-back map $f^* : G(Z) \to G(Z')$ sending the class of a coherent O_Z-module $\mathcal{F}$ to the class of the coherent complex $Lf^*\mathcal{F} = O_X \otimes^L_{O_Y} \mathcal{F}$. The maps f_* and f^* preserve topological filtrations in the following way.

Lemma 2.2 *Let $f : X \to Y$ be a morphism of schemes of finite type over a regular noetherian scheme S of finite dimension.*

1. ([9] Chapter VI Proposition 5.6) *If f is proper, we have $f_*F_nG(X) \subset F_nG(Y)$.*

2. ([9] Chapter VI Proposition 6.3) *If f is flat of relative dimension m, we have $f^*F_nG(Y) \subset F_{n+m}G(X)$.*

3. (loc.cit.) *Assume f is a regular immersion of codimension c. For a subscheme Z of Y and $Z' = Z \times_Y X$, we have $f^*F_nG(Z) \subset F_{n-c}G(Z')$.*

Corollary 2.3 *Assume further that X and Y are regular, S is affine, X is quasi-projective over S, and $f : X \to Y$ is proper, surjective and generically finite. Let $Z \subset Y$ be a subscheme and put $Z' = Z \times_Y X$. Then,* *1. The map $f^*: G(Z) \to G(Z')$ sends $F_nG(Z)$ into $F_nG(Z')$.*

2. The composition $f_*f^*: Gr^F_n G(Z) \to Gr^F_n G(Z)$ is the multiplication by $[X:Y]$.

Proof of Corollary. 1. Take an immersion $X \to \mathbf{P}^N_S$. The map $X \to Y$ is factorized as $X \to \mathbf{P}^N_S \times_S Y \to Y$. Since X and Y are regular, the immersion $X \to \mathbf{P}^N_S \times_S Y$ is regular of codimension N. Hence it follows from Lemma 2.2.2 and 3.

2. The direct image Rf_*O_X is a perfect complex of rank $[X:Y]$. For a coherent O_Z-module $\mathcal{F}$ such that $\dim\ \mathrm{supp}\mathcal{F} = n$, the class $Rf_*Lf^*\mathcal{F} = \mathcal{F} \otimes_{O_Y} Rf_*O_X$ is equal to $[X:Y][\mathcal{F}]$ modulo $F_{n-1}G(Z)$.

The filtrations on K-groups and Chow groups are related as follows. Let X be a scheme of finite type over a regular noetherian scheme S of finite dimension. Let $CH^*(X)$ denotes the bivariant Chow ring $CH^*(X \to X)$. If X is of dimension d, there is a canonical map $\cap[X]: CH^q(X) \to CH_{d-q}(X)$. It is an isomorphism if X is smooth over a field, [8] Corollary 17.4. The map $ch: K(X) \to CH^*(X)_{\mathbf{Q}}$ sending the class $[\mathcal{E}]$ of a locally free O_X-module $\mathcal{E}$ to its chern character $(ch_i(\mathcal{E}))_i \in CH^*(X)_{\mathbf{Q}}$ is a ring homomorphism.

Lemma 2.4 *Let X be a scheme of finite type over a regular noetherian scheme S of finite dimension.*

1. The chern character map $ch: K(X) \to CH^*(X)_{\mathbf{Q}}$ is compatible with the γ-filtration and induces a homomorphism $ch: Gr^*_F K(X) \to CH^*(X)_{\mathbf{Q}}$ of graded rings.

2. (cf. [8] Example 15.1.5) *The map $CH_*(X) \to Gr^F_* G(X)$ sending the class $[V]$ of integral subscheme V to $[O_V]$ is well-defined and is a surjection.*

3. Assume X is purely of dimension n. Then the composition

$$Gr^*_F K(X)_{\mathbf{Q}} \xrightarrow{ch} CH^*(X)_{\mathbf{Q}} \xrightarrow{\cap[X]} CH_{n-*}(X)_{\mathbf{Q}} \longrightarrow Gr^F_{n-*} G(X)_{\mathbf{Q}}$$

is equal to the map induced by the canonical map $K(X) \to G(X)$.

4. Assume X is quasi-projective and smooth of dimension n over a field. Then the three maps in 3 are isomorphisms.

By Lemma 2.4, the intersection product on $CH_*(X)_{\mathbf{Q}}$ for a smooth quasi-projective scheme X over a field may be computed by the product on $K(X)_{\mathbf{Q}}$.

Proof. 1. It follows from the splitting principle and the equality $\gamma_t([\mathcal{L}]-1) = 1 + ([\mathcal{L}]-1)t$ for an invertible sheaf $\mathcal{L}$.

2. Let W be a closed subscheme of $\mathbf{P}^1_X$ and let $\pi: \mathbf{P}^1_X \to X$ be the projection. Then we have $[O_{W_0}] - [O_{W_\infty}] = \pi_*(([O(1) - O] - [O(1) - O]) \cdot [O_W]) = 0$ in $G(X)$.

3. It follows from the splitting principle and the equality $ch_1([O(D)] - 1) \cap [X] = [D]$ for a Cartier divisor D.

4. The second arrow is an isomorphism by [8] Corollary 17.4. The composition is an isomorphism by 3 and by [9] Chapter VI Proposition 5.5. By Riemann-Roch for the immersion $V \to X$, we have $ch_i[O_V] = [V]$ for a closed subscheme V of codimension i. Hence the composition map $CH_{n-i}(X)_{\mathbf{Q}} \to Gr^F_{n-i} G(X)_{\mathbf{Q}} \to Gr^i_F K(X)_{\mathbf{Q}} \to CH_{n-i}(X)_{\mathbf{Q}}$ is the identity. Thus the assertion follows.

The following is an immediate consequence of the splitting principle and the equality (4).

Lemma 2.5 *For a locally free sheaf $\mathcal{E}$ of rank n, the class of $(-1)^n\gamma_n([\mathcal{E}]-n) = \sum_q(-1)^q[\Lambda^q\mathcal{E}]$ in $Gr_n^F G(X)$ is equal to the image of $c_n(\mathcal{E})$.*

Using this Lemma, we prove a K-theoretic version of the excess intersection formula. We will show its localized version, Proposition 2.16, later in this section. Let

$$\begin{array}{ccc} T & \xrightarrow{i'} & W \\ {\scriptstyle\varphi_T}\downarrow & & \downarrow{\scriptstyle\varphi} \\ V & \xrightarrow[i]{} & X \end{array}$$

be a cartesian diagram of noetherian schemes. Assume i is a regular immersion of codimension c and i' is a regular immersion of codimension c'. The excess conormal sheaf $N'_{V/X,W}$ is defined to be the kernel $\mathrm{Ker}(\varphi_T^* N_{V/X} \to N_{T/W})$ of the canonical surjection of conormal sheaves. It is a locally free O_T-module of rank $c-c'$.

Lemma 2.6 *Let*

$$\begin{array}{ccc} T & \xrightarrow{i'} & W \\ {\scriptstyle\varphi_T}\downarrow & & \downarrow{\scriptstyle\varphi} \\ V & \xrightarrow[i]{} & X \end{array}$$

be a cartesian diagram of noetherian schemes. Assume i is a regular immersion of codimension c and i' is a regular immersion of codimension c'. Then,

1. There is a canonical isomorphism $\Lambda^q N'_{V/X,W} \to \mathcal{T}or_q^{O_X}(O_V, O_W)$ of O_T-modules.

2. For the intersection product $(V,W)_X = \sum_q(-1)^q[\mathcal{T}or_q^{O_X}(O_V,O_W)] \in G(T)$, we have an equality

$$(V,W)_X = \sum_q (-1)^q[\Lambda^q N'_{V/X,W}] = (-1)^{c-c'}\gamma_{c-c'}([N'_{V/X,W}] - (c-c')).$$

If W is pure of dimension p, it is in the topological filtration $F_{p-c}G(T)$ and its image in $Gr_{p-c}^F G(T)$ is equal to the image of the top chern class $(-1)^{c-c'}c_{c-c'}(N'_{V/X,W}) \cap [T]$.

Proof. 1. Let $I_V \subset O_X$ be the ideal of V and $I_T \subset O_W$ be the ideal of T. Then by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}or_1^{O_X}(O_V,O_W) & \longrightarrow & \varphi^* I_V & \longrightarrow & I_T & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N'_{V/X,W} & \longrightarrow & \varphi_T^* N_{V/X} & \longrightarrow & N_{T/W} & \longrightarrow & 0 \end{array}$$

of exact sequences, we obtain a canonical map $\mathcal{T}or_1^{O_X}(O_V,O_W) \to N'_{V/X,W}$. We show that it is an isomorphism and that the inverse induces isomorphisms $\Lambda^q N'_{V/X,W} \to \mathcal{T}or_q^{O_X}(O_V,O_W)$ by product. The question is local on W. Hence we may assume I_V is generated by $a=(a_1,\ldots,a_c)$ and I_T is generated by $b=(b_1,\ldots,b_{c'})$. Let $\mathcal{K} = \mathrm{Kos}(\mathcal{E} \xrightarrow{a} O_X) \to O_V$ be a Koszul resolution of O_V and $\mathcal{K}' = \mathrm{Kos}(\mathcal{E}' \xrightarrow{b} O_W) \to O_T$ be a Koszul resolution of O_T. We take a surjection $\varphi^*\mathcal{E} \to \mathcal{E}'$ compatible with the surjections to I_T by shrinking further W if necessary. The kernel $\mathcal{F} = \mathrm{Ker}(\varphi^*\mathcal{E} \to \mathcal{E}')$ is a locally free O_W-module of rank $c - c'$ and $\mathcal{F}|_T$ is canonically isomorphic to the excess conormal sheaf $N'_{V/X,W}$. We define a decreasing filtration on $\Lambda^q\varphi^*\mathcal{E}$ by $F^p(\Lambda^q\varphi^*\mathcal{E}) = \mathrm{Im}(\Lambda^p\mathcal{F} \otimes (\Lambda^{q-p}\varphi^*\mathcal{E}) \to \Lambda^q\varphi^*\mathcal{E})$. Then it induces a filtration F on the Koszul complex $\mathcal{K}_W = L\varphi^*\mathcal{K} = \mathrm{Kos}(\varphi^*\mathcal{E} \xrightarrow{a} O_W)$. The graded complex $Gr^p_F L\varphi^*\mathcal{K}$ is equal to $\Lambda^p\mathcal{F} \otimes \mathcal{K}'[p]$ and hence quasi-isomorphic to $\Lambda^p\mathcal{F} \otimes_{O_W} O_T[p] = \Lambda^p N'_{V/X,W}[p]$. Since the filtration F on the complex $\mathcal{K}_W$ is locally splitting, we have an isomorphism $\mathcal{T}or_p^{O_X}(O_V, O_W) \to \mathcal{H}_p\mathcal{K}_W \to \Lambda^p N'_{V/X,W}$. If $p = 1$, it is the same as the map defined in the beginning of the proof. Since the isomorphism is compatible with the product, the assertion is proved.

2. It follows immediately from 1 and Lemma 2.5.

For a scheme over a discrete valuation ring, we have a reduction map.

Lemma 2.7 *Let X be a scheme of finite type over a discrete valuation ring $S = \mathrm{Spec}\ O_K$. Then*

1. The map $(\ ,s)_S : G(X) \to G(X_s)$ is factored by the canonical surjection $G(X) \to G(X_K)$. We let $(\ ,s)_S : G(X_K) \to G(X_s)$ denote the induced map.

2. The induced map $(\ ,s)_S : G(X_K) \to G(X_s)$ sends $F_nG(X_K)$ into $F_nG(X_s)$.

Proof. We have an exact sequence $G(X_s) \to G(X) \to G(X_K) \to 0$. It is sufficient to show that the composition $G(X_s) \to G(X) \to G(X_s)$ is the 0-map. For a coherent O_{X_s}-module $\mathcal{F}$, we have $\mathcal{T}or_i^{O_K}(\mathcal{F}, F) \simeq \mathcal{F}$ for $i = 0, 1$ and 0 for otherwise. Hence we have $([\mathcal{F}], s) = [\mathcal{T}or_0^{O_K}(\mathcal{F}, F)] - [\mathcal{T}or_1^{O_K}(\mathcal{F}, F)] = [\mathcal{F}] - [\mathcal{F}] = 0$.

2. Let V be an integral closed subscheme of dimension n of X_K. Then the closure $\bar{V}$ with reduced scheme structure is flat over $S = \mathrm{Spec}\ O_K$ of relative dimension n. Hence we have $(V, s)_S = [O_{\bar{V}} \otimes_{O_K} F] \in F_nG(X_s)$.

2.2. Localized K-theoretic intersection product.

We state the theorem claiming the existence of the localized K-theoretic intersection product and some basic properties of it. The definition of localized K-theoretic intersection product is given in Definition 2.13 using Theorem 2.10. A comparison with the usual intersection product is given in Proposition 2.14. The proofs of statements will be given in the following subsections.

Definition 2.8 *We say a scheme X over S is locally a hypersurface of relative dimension $n-1$ if, for each $x \in X$, there exist an open neighborhood U of x in X and a regular immersion $U \to P$ of codimension 1 over S into a smooth scheme P over S of relative dimension n.*

Clearly, a scheme is locally of complete intersection if it is locally a hypersurface.

Example. Let K be a discrete valuation field with perfect residue field and X be a regular flat scheme of dimension n of finite type over O_K with smooth generic fiber. Then Lemma 1.17 means that X is locally a hypersurface of relative dimension $n-1$ over $S = \mathrm{Spec}\ O_K$.

Lemma 2.9 *Let X be a scheme over S which is locally a hypersurface of relative dimension $n-1$. Regard the closed subset*

$$Z = \{x \in X : X \textit{ is not smooth at } x \textit{ over } S\}$$

as a closed subscheme of X defined by the annihilator ideal $I_Z = \mathrm{Ann}\ \Omega^n_{X/S}$ and let $i : Z \to X$ be the immersion. Then $\mathcal{L}_Z = L^1 i^* L_{X/S}$ is an invertible O_Z-module.

Proof. The assertion is local on X. Shrinking X, we take a regular immersion $X \to P$ of codimension 1 over S into a smooth scheme P over S. Shrinking further, we take trivializations $\Omega^1_{P/S}|_X \simeq O_X^n$ and $N_{X/P} \simeq O_X$. Then the map $d : N_{X/P} \to \Omega^1_{P/S}|_X$ is defined by a vector $(a_1, \ldots, a_n) \in O_X^n$. The closed subscheme Z is defined by the ideal $(a_1, \ldots, a_n)$ and we have a canonical isomorphism $\mathcal{L}_Z \to N_{X/P}|_Z$.

In the following, we keep the notation in Lemma 2.9. Further, for a scheme W over X, we put $Z_W = Z \times_X W$. If further W is noetherian, let $G(Z_W)_{/\mathcal{L}_Z}$ denote the cokernel of the endomorphism $[\mathcal{G}] \mapsto [\mathcal{G}] - [\mathcal{G} \otimes_{O_Z} \mathcal{L}_Z]$ of $G(Z_W)$.

Theorem 2.10 *Let S be a scheme and X be a scheme over S which is locally a hypersurface over S of relative dimension $n-1$. Let V be a closed subscheme of X, W be a scheme over X, $\mathcal{F}$ be an O_V-module and $\mathcal{G}$ be a complex of O_W-modules. Assume that $\mathcal{F}$ is of tor-dimension $\leq m$ as an O_S-module and that $\mathcal{H}_q(\mathcal{G}) = 0$ except for $a \leq q \leq b$. We put $q_0 = m + n + b$. Then,*

1. *The O_{W_V}-module $\mathcal{T}or_q^{O_X}(\mathcal{F}, \mathcal{G})$ is an $O_{Z_{W_V}}$-module for $q \geq q_0$.*
2. *There exists a canonical isomorphism*

$$\iota = \iota_{\mathcal{F},\mathcal{G},X/S} : \mathcal{T}or_q^{O_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{T}or_{q-2}^{O_X}(\mathcal{F}, \mathcal{G}) \otimes_{O_Z} \mathcal{L}_Z$$

for $q - 2 \geq q_0$, which is functorial in $\mathcal{F}$ and $\mathcal{G}$.

Theorem 2.10.2 has the following consequence.

Corollary 2.11 *Assume further that S and W are noetherian, $\mathcal{F}$ is a coherent O_V-module and that $\mathcal{G}$ is a coherent complex of O_W-modules. Then, for $q \geq q_0$, the $O_{Z_{W_V}}$-module $\mathcal{T}or_q^{O_X}(\mathcal{F}, \mathcal{G})$ is coherent and the class $[\mathcal{T}or_q^{O_X}(\mathcal{F}, \mathcal{G})] \in G(Z_{W_V})_{/\mathcal{L}_Z}$ depends only on the parity $q \bmod 2$.*

We put $[\mathcal{T}or_{\mathrm{even}}^{O_X}(\mathcal{F}, \mathcal{G})] = [\mathcal{T}or_q^{O_X}(\mathcal{F}, \mathcal{G})]$ for $q \geq q_0$ and even, $[\mathcal{T}or_{\mathrm{odd}}^{O_X}(\mathcal{F}, \mathcal{G})] = [\mathcal{T}or_q^{O_X}(\mathcal{F}, \mathcal{G})]$ for $q \geq q_0$ and odd and

$$[[\mathcal{F}, \mathcal{G}]]_X = [\mathcal{T}or_{\mathrm{even}}^{O_X}(\mathcal{F}, \mathcal{G})] - [\mathcal{T}or_{\mathrm{odd}}^{O_X}(\mathcal{F}, \mathcal{G})].$$

If W is also a closed subscheme of X, $\mathcal{G}$ is a coherent O_W-module and if $\mathcal{G}$ is of finite tor-dimension as an O_S-module, we have $[[\mathcal{F}, \mathcal{G}]]_X = [[\mathcal{G}, \mathcal{F}]]_X$.

Lemma 2.12 *We keep the notation above.*

1. For exact sequences $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ and $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ of modules, we have

$$[[\mathcal{F}, \mathcal{G}]] = [[\mathcal{F}', \mathcal{G}]] + [[\mathcal{F}'', \mathcal{G}]] \quad \text{and} \quad [[\mathcal{F}, \mathcal{G}]] = [[\mathcal{F}, \mathcal{G}']] + [[\mathcal{F}, \mathcal{G}'']].$$

2. Let F be an increasing filtration on a coherent complex $\mathcal{G}$ of O_W-modules. Assume that $F_q\mathcal{G}$ is acyclic for sufficiently small q, $\mathcal{G}/F_q\mathcal{G}$ is acyclic for sufficiently large q and that $\mathrm{Gr}^F_q \mathcal{G}$ are coherent for all q. Then we have

$$[[\mathcal{F}, \mathcal{G}]]_X = \sum_q [[\mathcal{F}, \mathrm{Gr}^F_q \mathcal{G}]]_X.$$

Definition 2.13 *Let X be locally a hypersurface of relative dimension $n-1$ over a regular noetherian scheme S of finite dimension, $i : Z \to X$ be the immersion defined by the ideal* Ann $\Omega^n_{X/S}$ *and $\mathcal{L}_Z = L^1 i^* \Omega^1_{X/S}$. For a closed subscheme V of X and a noetherian scheme W over X, we call the bilinear map*

$$[[\ ,\ ]]_X : G(V) \times G(W) \to G(Z_{W_V})_{/\mathcal{L}_Z}$$

sending $(\mathcal{F}, \mathcal{G})$ to

$$[[\mathcal{F}, \mathcal{G}]]_X = [\mathcal{T}or^{O_X}_{\text{even}}(\mathcal{F}, \mathcal{G})] - [\mathcal{T}or^{O_X}_{\text{odd}}(\mathcal{F}, \mathcal{G})]$$

the localized intersection product on X. We put $[[V, W]]_X = [[O_V, O_W]]_X$.

Example. 1. Let G be a finite cyclic group of order m. We take $S = \text{Spec } \mathbf{Z}$ and $X = \text{Spec } \mathbf{Z}[G]$ where $\mathbf{Z}[G]$ denotes the group algebra. Since $X = \mu_m = \text{Spec } \mathbf{Z}[T]/(T^m - 1)$ is a divisor of $\mathbf{A}^1_{\mathbf{Z}}$, it is a hypersurface over S of relative dimension 0. The closed subscheme $Z \subset X$ is defined by the ideal (m) and the invertible O_Z-module $\mathcal{L}_Z$ is trivial. We apply Theorem 2.10 to $V = \text{Spec } \mathbf{Z}$ regarded as a subscheme of X by the augmentation $\mathbf{Z}[G] \to \mathbf{Z}$. Then Theorem 2.10 implies that, for a G-module M, there is an isomorphism $Tor^{\mathbf{Z}[G]}_q(\mathbf{Z}, M) \to Tor^{\mathbf{Z}[G]}_{q-2}(\mathbf{Z}, M)$ for $q - 2 > 0$. Since $Tor^{\mathbf{Z}[G]}_q(\mathbf{Z}, M)$ is the same as the homology group $H_q(G, M)$, it is the same as the well-known periodicity of the homology of cyclic group. If we identify $G(Z_V) = G(\mathbf{Z}/m\mathbf{Z}) \simeq \bigoplus_{p|m} \mathbf{Z}$ with the subgroup of $\mathbf{Q}^\times$ generated by the prime divisors of m, the product $[[\mathbf{Z}, M]]_{\text{Spec}\mathbf{Z}[G]} \in \mathbf{Q}^\times$ is the same as the Herbrand quotient $\#\hat{H}_0(G, M)/\#H_1(G, M)$.

For an arbitrary regular noetherian scheme S', the intersection product $[[S',]]_{\mu_{m,S'}} : G(\mu_{m,S'}) \to G(Z_{S'})$ is defined. Here S' is regarded as a subscheme of $\mu_{m,S'}$ by the unit-section and $Z_{S'}$ is the closed subscheme of S' defined by the ideal (m). We will show in Lemma 3.23 that the composition $[[S',]]_{\mu_{m,S'}} : G(\mu_{m,S'}) \to G(Z_{S'}) \to G(S')$ is the 0-map.

2. Let K be a discrete valuation field with perfect residue field and X be a regular flat scheme of finite type over $S = \text{Spec } O_K$ of dimension n with smooth generic fiber. Since X is flat and locally a hypersurface of relative dimension $n-1$ over S, the projection $pr_2 : X \times_S X \to X$ is also locally a hypersurface of relative dimension $n-1$ over X. Let $i : Z \to X$ be the closed subscheme defined by the ideal Ann $\Omega^n_{X/S}$ and $\mathcal{L}_Z$ be the invertible O_Z-module $L^1 i^* \Omega^1_{X/S}$. Let $\tilde{i} : \tilde{Z} \to X \times_S X$ be the closed subscheme defined by the ideal Ann $\Omega^n_{X \times_S X/X}$ and $\tilde{\mathcal{L}}_{\tilde{Z}}$ be the invertible $O_{\tilde{Z}}$-module $L^1 \tilde{i}^* \Omega^1_{X \times_S X/X}$. We regard X as a subscheme of $X \times_S X$ by the diagonal map. Then we have $Z = \tilde{Z} \times_{X \times_S X} X$ and $\mathcal{L}_Z = \tilde{\mathcal{L}}_{\tilde{Z}}|_Z$. We show that the map $[\mathcal{L}_Z] : G(Z) \to G(Z)$ is the identity. In fact, $G(Z)$ is generated by the classes $\pi_*[O_W] = [R\pi_* O_W]$ where W runs the normalization of a closed integral subscheme of Z and $\pi : W \to Z$ denotes the natural map. By Lemma 1.25, we have $[\mathcal{L}_Z] \cdot [R\pi_* O_W] = [R\pi_* \pi^* \mathcal{L}_Z] = [R\pi_* O_W]$ and the assertion follows. Therefore we have $G(Z)_{/\mathcal{L}_Z} = G(Z)$. Since the generic fiber is smooth, the subscheme Z is supported on the closed fiber X_s. Hence the localized intersection product with the diagonal defines a homomorphism

$$[[\ , X]]_{X \times_S X} : G(X \times_S X) \longrightarrow G(Z) \longrightarrow G(X_s).$$

We show in Corollary 2.18 that the self-intersection product $[[X, X]]_{X \times_S X} \in G(X_s)$ is equal to the image of the self-intersection class $(\Delta_X, \Delta_X)_S \in CH_0(X_s)$ defined in §1.3. In Section 3 Definition 3.18, we will define a logarithmic version of this product.

The localized product is related to the usual intersection product in the following way.

Proposition 2.14 *Let X be locally a hypersurface over a regular noetherian scheme S and V be a closed subscheme of X. Let P be a smooth scheme over S and $X \to P$ be a regular immersion of codimension 1. For a noetherian scheme W over X, the canonical map $G(Z_{W_V}) \to G(W_V)$ induces a map $G(Z_{W_V})_{/\mathcal{L}_Z} \to G(W_V)_{/N_{X/P}}$. Further we have a commutative diagram*

$$\begin{array}{ccc} [[\ ,\ ]]_X : G(V) \times G(W) & \longrightarrow & G(Z_{W_V})_{/\mathcal{L}_Z} \\ \| & & \downarrow \\ (\ ,\ )_P : G(V) \times G(W) & \longrightarrow & G(W_V)_{/N_{X/P}}. \end{array}$$

When a closed subscheme $V \subset X$ is locally of complete intersection over S, we prove an excess intersection formula, Proposition 2.16, for the map $[[V,\ ]]_X : G(X) \to G(Z_V)_{/\mathcal{L}_Z}$. It is a localized version of Lemma 2.6. To state it, we recall the conormal complex of an immersion. Let X and V be schemes locally of complete intersection over a scheme S and let $j : V \to X$ be an immersion over S. The conormal complex $M_{V/X}$ is defined to be the mapping fiber of the canonical map $Lj^*L_{X/S} \to L_{V/S}$. It is locally described as follows. Let U be an open subscheme of X and $U \to P$ be a regular immersion into a smooth scheme P over S. Then the immersion $V_U = V \cap U \to U \to P$ is also regular and the restriction $M_{V/X}|_{V_U}$ is quasi-isomorphic to the complex $[N_{U/P}|_{V_U} \to N_{V_U/P}]$ where $N_{V_U/P}$ is put on degree 0.

Lemma 2.15 *Let X and V be schemes locally of complete intersection over a scheme S and let $j : V \to X$ be an immersion over S. Let $M_{V/X} = \mathrm{Fiber}(Lj^*L_{X/S} \to L_{V/S})$ be the conormal complex. Then,*

1. The cohomology sheaf $\mathcal{H}_0(M_{V/X})$ is equal to the conormal sheaf $N_{V/X}$.

2. Assume there exists a dense open subscheme $W \to V$ where the immersion $W \to X$ is regular. Either if $W = V$ or if V is reduced, the complex $M_{V/X}$ is quasi-isomorphic to $N_{V/X}$.

3. If $j : V = S \to X$ is a section, we have $M_{V/X} = Lj^*L_{X/S}$.

4. Let W be a subscheme of X containing V as a subscheme and assume W is locally of complete interesectin over S. If $i : V \to W$ denotes the immersion, we have a distinguished triangle

$$Li^*M_{W/X} \longrightarrow M_{V/X} \longrightarrow M_{V/W} \longrightarrow .$$

Proof. 1. It follows immediately from the local description above.

2. If $W = V$, it follows from Lemma 1.24. The rest follows from the local description above.

3. Clear from the definition.

4. Follows from Lemma 1.24.

We also introduce excess conormal complex. Let $V \subset X \to S$ be as in Lemma 2.15. For a scheme W over X such that the immersion $T = V \times_X W \to W$ is a regular immersion, we define the excess conormal complex $M'_{V/X,W}$ to be the mapping fiber $\mathrm{Fiber}[L\varphi_T^* M_{V/X} \to N_{T/W}]$ of complexes of O_T-modules where $\varphi_T : T \to V$ is the natural map. If $W = T$ is a scheme over V, we have $M'_{V/X,W} = \varphi_T^* M_{V/X}$. If $X \to P$ is a regular immersion into a smooth scheme over S, the complex $M'_{V/X,W}$ is quasi-isomorphic to $[\varphi_T^* N_{X/P}|_V \to N'_{V/P,W}]$ where the excess conormal sheaf $N'_{V/P,W} = \mathrm{Ker}(\varphi_T^* N_{V/P} \to N_{T/W})$ is put on degree 0.

Let X be locally a hypersurface of relative dimension $n-1$ over a scheme S and V be a closed subscheme of X which is locally of complete intersection over S of relative dimension $n-c$. Let Z be the closed subscheme of X defined by the ideal Ann $\Omega^n_{X/S}$. On the complement $V - Z_V$ of $Z_V = V \cap Z$, the immersion $V - Z_V \to X - Z$ is regular of codimension $c-1$. Hence the restriction $M_{V/X}|_{V-Z_V}$ of the conormal complex $M_{V/X}$ is quasi-isomorphic to the conormal sheaf $N_{V-Z_V/X-Z}$ which is locally free of rank $c-1$. Therefore, if there exists a map $\mathcal{L} \to \mathcal{E}$ of locally free O_V-modules of finite rank and a quasi-isomorphism $[\mathcal{L} \to \mathcal{E}] \to M_{V/X}$, the localized chern class $c_c{}^V_{Z_V}(M_{V/X}) \in CH^c(Z_V \to V)$ is defined. Let W be a scheme over X such that the immersion $T = V \times_X W \to W$ is a regular immersion of codimension c'. On the complement $T - Z_T$, the excess conormal complex $M'_{V/X,W}|_{T-Z_T}$ is quasi-isomorphic to the excess conormal sheaf $N'_{V-Z_V/X-Z,W-Z_W}$ which is a locally free O_{T-Z_T}-module of rank $c-c'-1$. Further, if $[\mathcal{L} \to \mathcal{E}] \to M_{V/X}$ is a quasi-isomorphism of complexes of O_V-modules, the map $[\varphi_T^*\mathcal{L}|_V \to \mathrm{Ker}(\varphi_T^*\mathcal{E}|_V \to N_{T/W})] \to M'_{V/X,W}$ is a quasi-isomorphism. Hence $c_{c-c'}{}^T_{Z_T}(M'_{V/X,T}) \in CH^{c-c'}(Z_T \to T)$ is defined.

Let $F_pG(Z_V)_{/\mathcal{L}_Z}$ denote the filtration on $G(Z_V)_{/\mathcal{L}_Z}$ induced by the topological filtration on $G(Z_V)$.

Proposition 2.16 *Let X be locally a hypersurface of relative dimension $n-1$ over a noetherian scheme S and V be a closed subscheme of X. Assume that V is locally of complete intersection over S of relative dimension $n-c$ and that there exist a map $\mathcal{L} \to \mathcal{E}$ of locally free O_V-modules of finite rank and a quasi-isomorphism $[\mathcal{L} \to \mathcal{E}] \to M_{V/X}$. Let W be a scheme over X and assume W is of finite type over a regular noetherian scheme of finite dimension.*

1. The map $[[V,\ ]]_X : G(W) \to G(Z_{W_V})_{/\mathcal{L}_Z}$ sends the topological filtration $F_pG(W)$ to $F_{p-c}G(Z_{W_V})_{/\mathcal{L}_Z}$ for $p \geq 0$.

2. Assume W is of dimension p and that the immersion $T = V \times_X W \to W$ is a regular immersion of codimension c'. Then the class of $[[V,W]]_X \in F_{p-c}(Z_T)_{/\mathcal{L}_Z}$ in $Gr^F_{p-c}G(Z_T)_{/\mathcal{L}_Z}$ is equal to the image of $(-1)^{c-c'}c_{c-c'}{}^T_{Z_T}(M'_{V/X,W}) \cap [T] \in CH_{p-c}(Z_T)$.

We call the equality in 2 the excess intersection formula. Proof of Proposition 2.16 will be given in §2.5. In §2.4, we introduce an exterior power complex $L\Lambda^{c-c'}M'_{V/X,W}$ and relate it to the localized chern class $(-1)^{c-c'}c_{c-c'}{}^T_{Z_T}(M'_{V/X,W}) \cap [T]$ in Proposition 2.35. It is a localized version of Lemma 2.5. In §2.5, we compute $[[V,W]]_X$ in terms of the exterior power complex $L\Lambda^{c-c'}M'_{V/X,W}$ in Proposition 2.40 and complete the proof of Proposition 2.16. Proposition 2.40 is a localized version of Lemma 2.6.

Corollary 2.17 *If $W = T$ is a scheme over V, the class of $[[V,W]]_X \in F_{p-c}G(Z_W)_{/\mathcal{L}_Z}$ in $Gr^F_{p-c}G(Z_W)_{/\mathcal{L}_Z}$ is equal to the image of $(-1)^c c_c{}^V_{Z_V}(M_{V/X}) \cap [W] \in CH_{p-c}(Z_W)$. In particular, if $V = W$, we have a self-intersection formula*

$$[[V,V]]_X = (-1)^c c_c{}^V_{Z_V}(M_{V/X}) \cap [V]$$

in $Gr^F_{p-c}(Z_V)_{/\mathcal{L}_Z}$.

Proof of Corollary 2.17. If $W = T$, the excess conormal complex $M'_{V/X,W}$ is equal to the pull-back $\varphi_T M_{V/X}$. Hence the assertion follows.

As a special case, we have the following.

Corollary 2.18 *Let K be a discrete valuation field with perfect residue field and let X be a regular flat scheme of finite type of dimension n over O_K with smooth generic fiber. We regard X as a closed subscheme of $X \times_S X$ by the diagonal embedding $\Delta : X \to X \times_S X$. Then we have an equality*

$$[[X, X]]_{X \times_S X} = (-1)^n c_n{}_Z^X(\Omega^1_{X/S}) \cap [X] = (\Delta_X, \Delta_X)_S$$

in $Gr^F_0 G(X_s)$.

Proof of Corollary 2.18. Since X is flat over S, the projection $X \times_S X \to X$ is also locally a hypersurface of relative dimension $n-1$. Since $M_{X/X \times_S X} = L\Delta^* L_{X \times_S X/X} = L_{X/S} = \Omega^1_{X/S}$, it suffices to apply the self-intersection formula Proposition 2.17 to the diagonal embedding $\Delta : X \to X \times_S X$.

We show the compatibility of our K-theoretic localized intersection theory with the localized intersection theory in [1] using the excess intersection formula, Proposition 2.16, and the following projection formula.

Proposition 2.19 *Let X be locally a hypersurface of relative dimension $n-1$ over a noetherian scheme S. Let $\mathcal{F}$ be a coherent O_V-module on a closed subscheme V of X. Assume $\mathcal{F}$ is of finite tor-dimension as an O_S-module. Let $i : Z \to X$ be the closed subscheme of X defined by* $\operatorname{Ann} \Omega^n_{X/S}$ *and put $\mathcal{L}_Z = L^1 i^* \Omega^1_{X/S}$.*

1. Let $\pi : W' \to W$ be a proper morphism of noetherian schemes over X and $\mathcal{G}$ be a coherent complex of $O_{W'}$-modules. Then we have an equality

$$[[\mathcal{F}, R\pi_* \mathcal{G}]]_X = \pi_*[[\mathcal{F}, \mathcal{G}]]_X.$$

in $G(Z_W)_{/\mathcal{L}_Z}$

2. Let X' be another locally hypersurface of relative dimension $n'-1$ over a noetherian scheme S' and $f : X' \to X$ be a proper morphism of noetherian schemes. Let $\mathcal{G}$ be a coherent $O_{V'}$-module on a closed subscheme V' of X'. Assume $\mathcal{G}$ is of finite tor-dimension as an $O_{S'}$-module. Let $i' : Z' \to X'$ be the closed subscheme of X' defined by $\operatorname{Ann} \Omega^{n'}_{X'/S'}$ *and put $\mathcal{L}'_{Z'} = L^1 i'^* \Omega^1_{X'/S'}$. Let Z_1 be a closed subset of X containing the images of $Z_{V \times_X V'} = Z \times_X V \times_X V'$ and $Z'_{V \times_X V'} = V \times_X V' \times_{X'} Z'$ as subsets and let $G(Z_1)_{/\mathcal{L}_Z, \mathcal{L}'_{Z'}}$ be the cokernel of the map $(f_* \circ ([\mathcal{L}_Z] - 1), f_* \circ ([\mathcal{L}'_{Z'}] - 1)) : G(Z_{V \times_X V'}) \oplus G(Z'_{V \times_X V'}) \to G(Z_1)$. If the complex $Lf^* \mathcal{F}$ of $O_{X'}$-modules is coherent, we have an equality in $G(Z_1)_{/\mathcal{L}_Z, \mathcal{L}'_{Z'}}$*

$$[[\mathcal{F}, Rf_* \mathcal{G}]]_X = f_*[[\mathcal{G}, Lf^* \mathcal{F}]]_{X'}.$$

Let the assumption be as in Proposition 2.16. Let W be a scheme over X and assume W is of finite type over a regular noetherian scheme. For an integer $p \geq 0$, let $Z_p(W)$ be the free abelian group generated by the classes of integral closed subscheme of dimension p and let $(V, \ )_{\mathrm{loc}} : Z_p(W) \to CH_{p-c}(Z_{V_W})$ be the localized intersection product defined in [1] Definition 4.4.

Corollary 2.20 *Assume further that the conormal complex $M_{V/X}$ is quasi-isomorphic to the conormal sheaf $N_{V/X}$. Let W be a scheme over X and assume W is of finite type over a regular noetherian scheme. Then, for an integer $p \geq 0$, the diagram*

$$\begin{array}{ccc} Z_p(W) & \xrightarrow{(V,\ )_{\rm loc}} & CH_{p-c}(Z_{V_W}) \\ {\scriptstyle \rm can}\downarrow & & \downarrow{\scriptstyle \rm can} \\ Gr^F_p G(W) & \xrightarrow[{[[V,\ ]]_X}]{} & Gr^F_{p-c} G(Z_{V_W})_{/\mathcal{L}_Z} \end{array}$$

is commutative.

Proof. The localized intersection product $[[V,\ ]]_X$ is characterized by the excess intersection formula, Proposition 2.16, and the projection formula, Proposition 2.19. Similarly, $(V,\ )_{\rm loc}$ the localized intersection product is characterized by the localized excess intersection formula, [1] Proposition 4.11, and the projection formula loc.cit. Proposition 4.6 (a). Since the excess intersection formulas have the same form, the assertion follows.

Using excess intersection formula, we show that the image of the localized intersection product $[[V,W]]_X$ in $Gr^F_{p-c}G(T)_{/\mathcal{L}_Z}$ may be computed using the Segre classes. We briefly recall the definition of Segre class (cf. [8] Corollary 4.2.2). For a closed subscheme $T \neq W$ of W, its Segre class $s_i(T,W) \in CH_{p-i}(T)$ for $i > 0$ is defined as follows. Let $\pi : W' \to W$ be the blow-up at T and $D = \pi^{-1}(T) = W' \times_W T$ be the inverse image of T. The subscheme D is a Cartier divisor of W'. Then we define the i-th Segre class $s_i(T,W)$ to be $(-1)^{i-1}\pi_*(D^{i-1} \cap [D])$. We define the total Segre class by $s(T,W) = \sum_{i=1}^p s_i(T,W) \in \bigoplus_{i=1}^p CH_{p-i}(T)$. For a perfect complex $\mathcal{K}$, we put $c(\mathcal{K})^* = c(\mathcal{K}^*) = \sum_i (-1)^i c_i(\mathcal{K})$ as usual.

Corollary 2.21 *Let $V \subset X \to S$ and $T = V \times_X W \subset W \to X$ be as in Proposition 2.16. Assume W is an integral scheme of dimension p of finite type over a regular noetherian scheme of finite dimension and $T \neq W$. Let $\varphi_T : T \to V$ be the natural map, let $G(T)_{/\mathcal{L}_Z}$ denote the cokernel of the map $[\mathcal{L}_Z] - 1 : G(Z_T) \to G(T)$ and let $F_\bullet G(T)_{/\mathcal{L}_Z}$ denote the filtration induced by the topological filtration. Then the image of the localized intersection product $[[V,W]]_X$ in $Gr^F_{p-c}G(T)_{/\mathcal{L}_Z}$ is equal to the image of*

$$\{c(L\varphi_T^* M_{V/X})^* \cap s(T,W)\}_{\dim p-c} = \sum_{i=0}^{c-1} (-1)^i c_i(L\varphi_T^* M_{V/X}) s_{c-i}(T,W) \in CH_{p-c}(T).$$

Proof. Let $\pi : W' \to W$ be the blow-up at T and $D = \pi^{-1}(T) = W' \times_W T$ be the inverse image of T as above. By Proposition 2.19.1, we have $[[V,W]]_X = \pi_*[[V,W']]_X$. Since D is a Cartier divisor of W', by Proposition 2.16.2, the image of the localized product $[[V,W']]_X$ in $Gr^F_{p-c}G(D)_{/\mathcal{L}_Z}$ is equal to the image of $(-1)^{c-1}c_{c-1}(M'_{V/X,W'}) \cap [D] = \{c(M_{V/X})^* c(N_{D/W'})^{*-1} \cap [D]\}_{\dim p-c} \in CH_{p-c}(D)$. Hence $[[V,W]]_X$ is equal to the image of $\{c(L\varphi_T^* M_{V/X})^* \pi_*(c(N_{D/W'})^{*-1} \cap [D])\}_{\dim p-c}$. Since $N_{D/W'} = O(-D)|_D$, we have an equality $\pi_*(c(N_{D/W'})^{*-1} \cap [D]) = s(T,W)$. Thus we obtain the required equality.

As an application of the excess intersection formula, we give an interpretation of the Swan character as a localized intersection product.

Corollary 2.22 *Let K be a discrete valuation field with perfect residue field. Let $T = \text{Spec } O_L$ be the spectrum of the integer ring of a totally ramified finite separable extension L of K. Let t be the closed point of T and we identify $G(t) = \mathbf{Z}$. We keep the notation of §1.1. Then,*

1. We have

$$[[T, T]]_{(T\times_S T)^\sim} = -\text{length}_{O_T} N_{T/(T\times_S T)^\sim} \qquad \textit{and} \qquad [[T, T]]_{T\times_S T} = -\text{length}_{O_T} N_{T/T\times_S T}.$$

2. Assume L is a Galois extension of K. Then for an element $\sigma \in G_{L/K}$ of the Galois group, we have

$$\text{a}_{L/K}(\sigma) = -[[T, T_\sigma]]_{T\times_S T} \quad \text{and} \quad \text{sw}_{L/K}(\sigma) = -[[T, T_\sigma^\sim]]_{(T\times_S T)^\sim}$$

in $G(t) = \mathbf{Z}$. If $\sigma \in G_{L/K} - P_{L/K}$, the intersection $T_\sigma \cap T$ in $(T \times_S T)^\sim$ is empty.

Proof. In the proof of Lemma 1.2.2, we have shown that $T \times_S T$ and $(T \times_S T)^\sim$ are divisors of $P = \mathbf{A}^1_T$. Hence they are hypersurfaces over a regular scheme T.

1. We show the first equality by applying Proposition 2.16.2. We take T to be S in Proposition 2.16.2, $(T \times_S T)^\sim$ to be X and $\Delta^\sim : T \to (T \times_S T)^\sim$ to be $V = W$. Since T is regular, the conormal complex $M_{T/(T\times_S T)^\sim}$ is quasi-isomorphic to the conormal sheaf $N_{T/(T\times_S T)^\sim}$ by Lemma 2.15.2. Applying Proposition 2.16.2, we obtain $[[T, T]]_{(T\times_S T)^\sim} = -c1^T_{Z_T}(N_{T/(T\times_S T)^\sim}) \cap [T]$. Hence it follows from Corollary 1.6. The second equality is a special case of Corollary 2.18.

2. We may assume $\sigma \neq 1$. We further take $\Gamma^\sim_\sigma : T_\sigma \to (T \times_S T)^\sim$ to be W in Proposition 2.16.3. Then the intersection $D = T \times_{(T\times_S T)^\sim} T_\sigma$ is a divisor in T_σ. Hence by applying it, we obtain $[[T, T^\sim_\sigma]]_{(T\times_S T)^\sim} = \text{length}_{O_T} O_T \otimes_{O_{(T\times_S T)^\sim}} O_{T^\sim_\sigma}$. Similarly, we have $[[T, T_\sigma]]_{T\times_S T} = \text{length}_{O_T} O_T \otimes_{O_{T\times_S T}} O_{T_\sigma}$. Now it follows from Lemma 1.2.3 and 4.

The product satisfies associativity in the following sense.

Proposition 2.23 *Let X be locally a hypersurface over a noetherian scheme S of relative dimension $n - 1$ and $\mathcal{F}$ be a coherent O_V-module on a closed subscheme V of X. Assume $\mathcal{F}$ is of finite tor-dimension as an O_S-module. Let $\mathcal{G}$ be a coherent complex of O_W-modules on a noetherian scheme W over X. Let $i : Z \to X$ be the closed subscheme of X defined by $\text{Ann } \Omega^n_{X/S}$ and let $\mathcal{L}_Z$ be the invertible O_Z-module $L^1 i^*\Omega^1_{X/S}$.*

1. Let $W' \to W$ be a morphism of noetherian schemes and $\mathcal{H}$ be a coherent complex of $O_{W'}$-modules as in the diagram

$$\begin{array}{ccccc} \mathcal{F} & & \mathcal{G} & & \mathcal{H} \\ X & \longleftarrow & W & \longleftarrow & W'. \end{array}$$

Assume $\mathcal{H}$ is of finite tor-dimension as a complex of O_W-modules. Then we have an equality in $G(Z_{W'})_{/\mathcal{L}_Z}$

$$([[\mathcal{F}, \mathcal{G}]]_X, \mathcal{H})_W = [[\mathcal{F}, (\mathcal{G}, \mathcal{H})_W]]_X.$$

2. Let X' be locally a hypersurface over a noetherian scheme S' of relative dimension $n' - 1$, $\mathcal{F}'$ be a coherent $O_{V'}$-module on a closed subscheme V' of X' and let $W \to X'$ be a morphism of noetherian schemes as in the diagram

$$\begin{array}{ccccc} \mathcal{F} & & \mathcal{G} & & \mathcal{F}' \\ X & \longleftarrow & W & \longrightarrow & X'. \end{array}$$

Assume $\mathcal{F}'$ is of finite tor-dimension as an $O_{S'}$-module. Let $i' : Z' \to X'$ be the closed subscheme of X' defined by $\operatorname{Ann} \Omega^{n'}_{X'/S'}$ and let $\mathcal{L}'_{Z'}$ be the invertible $O_{Z'}$-module $L^1 i'^* \Omega^1_{X'/S'}$. Let Z_1 be a closed subset of W containing $Z_{W_V} = Z \times_X V \times_X W$ and $Z'_{W_{V'}} = Z' \times_{X'} V' \times_{X'} W$ as subsets and let $G(Z_1)_{/\mathcal{L}_Z, \mathcal{L}'_{Z'}}$ be the cokernel of the map $(\mathrm{can} \circ ([\mathcal{L}_Z] - 1), \mathrm{can} \circ ([\mathcal{L}'_{Z'}] - 1)) : G(Z_{W_V}) \oplus G(Z'_{W_{V'}}) \to G(Z_1)$. If the complex $\mathcal{G}$ is of finite tor-dimension as a complex of O_X-modules and as a complex of $O_{X'}$-modules, we have an equality in $G(Z_1)_{/\mathcal{L}_Z, \mathcal{L}'_{Z'}}$

$$[[\mathcal{F}, (\mathcal{F}', \mathcal{G})_{X'}]]_X = [[\mathcal{F}', (\mathcal{F}, \mathcal{G})_X]]_{X'}.$$

When a hypersurface is flat, the localized intersection product commutes with base change in the following sense.

Corollary 2.24 *Let X be locally a flat hypersurface over a noetherian scheme S of relative dimension $n-1$ and V be a closed subscheme of X. Let $f : T \to S$ be a morphism of finite tor-dimension of noetherian schemes. Let the closed subscheme $i : Z \to X$ the invertible O_Z-module $\mathcal{L}_Z = L^1 i^* \Omega^1_{X/S}$ be as in Proposition 2.23.*

1. Let $\mathcal{F}$ be a coherent O_V-module and $\mathcal{F}'$ be a coherent $O_{V'}$-module on a closed subscheme V' of $X_T = X \times_S T$. Assume $\mathcal{F}$ is of finite tor-dimension as an O_S-module and $\mathcal{F}'$ is of finite tor-dimension as an O_T-module. Then we have

$$[[\mathcal{F}, \mathcal{F}']]_X = [[\mathcal{F}', Lf^* \mathcal{F}]]_{X_T}$$

in $G(Z_{V \times_X V'})_{/\mathcal{L}_Z}$.

2. Assume V is flat over S and T is regular. Then we have a commutative diagram

$$\begin{array}{ccc} G(X) & \xrightarrow{[[V,\ ]]_X} & G(Z_{V_T})_{/\mathcal{L}_Z} \\ \uparrow & & \| \\ G(X_T) & \xrightarrow{[[V_T,\ ]]_{X_T}} & G(Z_{V_T})_{/\mathcal{L}_Z}. \end{array}$$

Proof of Corollary 2.24. 1. In Proposition 2.23.2, we take T to be S' in Proposition 2.23.2, X_T to be $X' = W$ and V' to be V'. We apply it to $\mathcal{G} = O_{X_T}$. Since X is flat and f is of finite tor-dimension, the O_X-module O_{X_T} is of finite tor-dimension and hence the assumption is satisfied. By applying it, we obtain the equality.

2. It is sufficient to show an equality $[[V, \mathcal{F}']]_X = [[V_T, \mathcal{F}']]_{X_T}$ for a coherent O_{X_T}-module $\mathcal{F}'$. We apply 1 to $\mathcal{F} = O_V, V' = X_T$ and $\mathcal{F}'$. Then $\mathcal{F} = O_V$ is flat as an O_S-module and $\mathcal{F}'$ is of finite tor-dimension as an O_T-module. Hence the assumption is satisfied. By applying 1, we obtain an equality $[[V, \mathcal{F}']]_X = [[\mathcal{F}', Lf^* O_V]]_{X_T}$ in $G(Z_{V_T})_{/\mathcal{L}_Z}$. Since V is flat over S, we have $Lf^* O_V = O_{V_T}$ and $[[\mathcal{F}', Lf^* O_V]]_{X_T} = [[V_T, \mathcal{F}']]_{X_T}$.

In the proof of conductor formula, we will use Proposition 2.23 in the following form.

Corollary 2.25 *Let S be a regular noetherian scheme and X be locally a hypersurface of finite type over S. Let $f : W \to X$ be a morphism of noetherian schemes.*

1. Let $g : W' \to W$ be a morphism of finite tor-dimension of noetherian schemes over X. Then the diagram

$$\begin{array}{ccc} G(X) & \xrightarrow{[[\ ,W]]_X} & G(Z_W)_{/\mathcal{L}_Z} \\ \| & & \downarrow g^* \\ G(X) & \xrightarrow[{[[\ ,W']]_X}]{} & G(Z_{W'})_{/\mathcal{L}_Z} \end{array}$$

is commutative.

2. Let $g : W' \to W$ be a morphism of noetherian scheme and V be a closed subscheme of X. Assume W is regular so that the map

$$(\ ,\ )_W : G(Z_{W_V})_{/\mathcal{L}_Z} \times G(W') \longrightarrow G(Z_{W'_V})_{/\mathcal{L}_Z}$$

sending $([\mathcal{F}], [\mathcal{H}])$ to $[\mathcal{F} \otimes^L_{O_W} \mathcal{H}]$ is well-defined. Then the map

$$[[V,\ ]]_X : G(W') \longrightarrow G(Z_{W'_V})_{/\mathcal{L}_Z}$$

is equal to the pairing

$$([[V, W]]_X,\ )_W : G(W') \longrightarrow G(Z_{W'_V})_{/\mathcal{L}_Z}$$

with $[[V, W]]_X \in G(Z_{W_V})_{/\mathcal{L}_Z}$.

3. Let S' be another regular noetherian scheme and X' be locally a hypersurface over S'. Let $g : W \to X'$ be a flat morphism, V' be a closed subscheme of X' and put $W' = W \times_{X'} V'$. Assume that $f : W \to X$ is a morphism of finite tor-dimension, that the closed subset $Z_{W'} = Z \times_X W'$ of W' is set-theoretically a subset of $Z'_{W'} = Z' \times_{X'} W'$ and that the natural map $G(Z_{W'}) \to G(Z'_{W'})$ induces a map $G(Z_{W'})_{/\mathcal{L}_Z} \to G(Z'_{W'})_{/\mathcal{L}'_{Z'}}$. Then the diagram

$$\begin{array}{ccc} G(X) & \xrightarrow{[[\ ,W']]_X} & G(Z_{W'})_{/\mathcal{L}_Z} \\ f^* \downarrow & & \downarrow \\ G(W) & \xrightarrow[{[[V',\ ]]_{X'}}]{} & G(Z'_{W'})_{/\mathcal{L}'_{Z'}} \end{array}$$

is commutative.

Proof of Corollary 2.25. 1. It is sufficient to show the equality $g^*[[\mathcal{F}, W]]_X = [[\mathcal{F}, W']]_X$ for a coherent O_X-module $\mathcal{F}$. This is a special case of Proposition 2.23.1 where $\mathcal{G} = O_W$ and $\mathcal{H} = O_{W'}$.

2. It is sufficient to show the equality $[[V, \mathcal{H}]]_X = ([[V, W]]_X, \mathcal{H})_W$ for a coherent $O_{W'}$-module $\mathcal{H}$. This is a special case of Proposition 2.23.1 where $\mathcal{F} = O_V$ and $\mathcal{G} = O_W$.

3. It is sufficient to show the equality $[[\mathcal{F}, W']]_X = [[V', Lf^*\mathcal{F}]]_{X_1}$ for a coherent O_X-module $\mathcal{F}$. By the flatness of $W \to X'$, we have $Lg^*O_{V'} = O_{W'}$. Hence this is a special case of Proposition 2.23.2 where $\mathcal{G} = O_W$ and $\mathcal{F}' = O_{V'}$.

2.3. Proofs.

In this subsection, we give proofs of the assertions in the previous subsection except for the excess intersection formula, Proposition 2.16.

To prove Theorem 2.10, we first define a canonical map

$$\iota = \iota_{\mathcal{F},X/P} : \mathcal{F} \to \mathcal{F} \otimes N_{X/P}[2]$$

in the derived category of O_X-modules for a regular immersion $i : X \to P$ of noetherian schemes and an O_X-module $\mathcal{F}$. To define it, we apply the following construction to the complex $\mathcal{K} = Li^*i_*\mathcal{F}$. For a complex $\mathcal{K}$ of O_X-modules, we define a canonical map

$$\iota : \mathcal{H}_0\mathcal{K} \to (\mathcal{H}_1\mathcal{K})[2]$$

in the derived category to be that defined by the exact sequence

$$0 \to \mathcal{H}_1\mathcal{K} \to \mathrm{Coker}(d_2 : \mathcal{K}_2 \to \mathcal{K}_1) \xrightarrow{d_1} \mathrm{Ker}(d_0 : \mathcal{K}_0 \to \mathcal{K}_{-1}) \to \mathcal{H}_0\mathcal{K} \to 0.$$

By Lemma 2.26 below, we identify $\mathcal{H}_0 Li^*i_*\mathcal{F} = \mathcal{F}$ and $\mathcal{H}_1 Li^*i_*\mathcal{F} = \mathcal{T}or_1^{O_P}(\mathcal{F}, O_X) = \mathcal{F} \otimes_{O_X} N_{X/P}$. Applying the above construction to $\mathcal{K} = Li^*i_*\mathcal{F}$, we define a canonical map $\iota_{\mathcal{F},X/P} : \mathcal{F} \to \mathcal{F} \otimes N_{X/P}[2]$.

Lemma 2.26 *Let $i : X \to P$ be a regular immersion and $\mathcal{F}$ be an O_X-module. Then there is a canonical isomorphism $\mathcal{T}or_q^{O_P}(\mathcal{F}, O_X) \to \mathcal{F} \otimes_{O_X} \Lambda^q N_{X/P}$.*

Proof. By Lemma 2.6, we have a canonical isomorphism $\mathcal{T}or_q^{O_P}(O_X, O_X) \to \Lambda^q N_{X/P}$. Hence the spectral sequence $E^2_{p,q} = \mathcal{T}or_q^{O_X}(\mathcal{F}, \mathcal{T}or_q^{O_P}(O_X, O_X)) \Rightarrow \mathcal{T}or_{p+q}^{O_P}(\mathcal{F}, O_X)$ degenerates at E^2-terms and gives the required isomorphism.

The canonical map $\iota_{\mathcal{F},X/P} : \mathcal{F} \to \mathcal{F} \otimes N_{X/P}[2]$ satisfies the following functorial property. The next lemma is an immediate consequence of the construction of the map ι and the map $Li^*i_*\mathcal{F}|_U \to Li'^*i'_*\mathcal{F}'$.

Lemma 2.27 *Let U be an open subscheme of X, $i' : U \to Q$ be a regular immersion and*

$$\begin{array}{ccc} U & \xrightarrow{i'} & Q \\ \cap\big\downarrow & & \big\downarrow \\ X & \xrightarrow[i]{} & P \end{array}$$

be a commutative diagram of schemes. For a morphism $f : \mathcal{F}|_U \to \mathcal{F}'$ of O_U-modules, the diagram

$$\begin{array}{ccc} \mathcal{F}|_U & \xrightarrow{\iota_{\mathcal{F},X/P}|_U} & \mathcal{F}|_U \otimes N_{U/P}[2] \\ f\big\downarrow & & \big\downarrow f\otimes\mathrm{can} \\ \mathcal{F}' & \xrightarrow[\iota_{\mathcal{F}',U/Q}]{} & \mathcal{F}' \otimes N_{U/Q}[2] \end{array}$$

is commutative.

For a complex $\mathcal{G}$ of O_W-modules on a scheme W over X, the map $\iota_{\mathcal{F},X/P} : \mathcal{F} \to \mathcal{F} \otimes N_{X/P}[2]$ induces a map $\iota_{\mathcal{F},\mathcal{G},X/P} : \mathcal{T}or_q^{O_X}(\mathcal{F},\mathcal{G}) \to \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P}$ of O_W-modules.

Corollary 2.28 *Let $X \to P$ be a regular immersion into a smooth scheme P over S and let $\mathcal{G}$ be a complex of O_W-modules on a scheme W over X. Then the composite*

$$\mathcal{T}or_q^{O_X}(\mathcal{F},\mathcal{G}) \xrightarrow{\iota_{\mathcal{F},\mathcal{G},X/P}} \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P} \xrightarrow{\mathrm{id}\otimes d} \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes \Omega^1_{P/S}|_X$$

is the 0-map.

Proof of Corollary. Let $p_1, p_2 : P \times_S P \to P$ denote the projections and $\Delta : P \to P \times_S P$ denote the diagonal. By Lemma 2.27, the composite

$$\mathcal{T}or_q^{O_X}(\mathcal{F},\mathcal{G}) \to \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P} \xrightarrow{p_1^*-p_2^*} \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P\times_S P}$$

is the zero map. The map $p_1^* - p_2^* : N_{X/P} \to N_{X/P\times_S P}$ is the composition of $d : N_{X/P} \to \Omega^1_{P/S}|_X$ with the inclusion $\Omega^1_{P/S}|_X \to N_{X/P\times_S P}$. By the exact sequence

$$0 \to \Omega^1_{P/S}|_X \to N_{X/P\times_S P} \xrightarrow{\Delta^*} N_{X/P} \to 0,$$

$\Omega^1_{P/S}|_X$ is locally a direct summand of $N_{X/P\times_S P}$. Hence the map $\mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes \Omega^1_{P/S}|_X \to \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P\times_S P}$ is injective. Thus the assertion follows.

Lemma 2.29 *Assume the immersion $X \to P$ is of codimension 1. Let $W \to X$ be a scheme over X and $\mathcal{G}$ be a complex of O_W-modules satisfying $\mathcal{T}or_q^{O_P}(\mathcal{F},\mathcal{G}) = 0$ for $q > q_0$. Then for $q-2 \geq q_0$, the map*

$$\iota_{\mathcal{F},\mathcal{G},X/P} : \mathcal{T}or_q^{O_X}(\mathcal{F},\mathcal{G}) \to \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P}$$

is an isomorphism.

To prove it, we apply the following elementary property of the map ι.

Lemma 2.30 *Let $\mathcal{K}$ be a complex of O_X-modules and $\mathcal{G}$ be a complex of O_W-modules on a scheme W over X. We consider the spectral sequence*

$$E^2_{p,q} = \mathcal{T}or_p^{O_X}(\mathcal{H}_q\mathcal{K},\mathcal{G}) \Rightarrow E_{p+q} = \mathcal{T}or_{p+q}^{O_X}(\mathcal{K},\mathcal{G}).$$

Then the map $\mathcal{T}or_p^{O_X}(\mathcal{H}_0\mathcal{K},\mathcal{G}) \to \mathcal{T}or_{p-2}^{O_X}(\mathcal{H}_1\mathcal{K},\mathcal{G})$ induced by $\iota : \mathcal{H}_0\mathcal{K} \to (\mathcal{H}_1\mathcal{K})[2]$ is the same as the degeneracy map $d^2_{p,0} : E^2_{p,0} = \mathcal{T}or_p^{O_X}(\mathcal{H}_0\mathcal{K},\mathcal{G}) \to E^2_{p-2,1} = \mathcal{T}or_{p-2}^{O_X}(\mathcal{H}_1\mathcal{K},\mathcal{G})$.

Proof of Lemma 2.29. We consider the spectral sequence

$$E^2_{p,q} = \mathcal{T}or_p^{O_X}(\mathcal{T}or_q^{O_P}(\mathcal{F},O_X),\mathcal{G}) \Rightarrow E_{p+q} = \mathcal{T}or_{p+q}^{O_P}(\mathcal{F},\mathcal{G}).$$

By Lemma 2.26, we have $E^2_{p,0} = \mathcal{T}or_p^{O_X}(\mathcal{F},\mathcal{G}), E^2_{p,1} = \mathcal{T}or_p^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P}$ and $E^2_{p,q} = 0$ unless $q \neq 0, 1$. Applying Lemma 2.30 to the complex $\mathcal{K} = \mathcal{F} \otimes^L_{O_P} O_X$, we see that the map $\iota_{\mathcal{F},\mathcal{G},X/P}$:

$\mathcal{T}or_p^{O_X}(\mathcal{F},\mathcal{G}) \to \mathcal{T}or_{p-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P}$ is the same as the degeneracy map $d_{p,0}^2 : \mathcal{T}or_p^{O_X}(\mathcal{F},\mathcal{G}) \to \mathcal{T}or_{p-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P}$. We show that the map $d_{p,0}^2 : E_{p,0}^2 \to E_{p-2,1}^2$ is an isomorphism. Since $E_{p,q}^2 = 0$ for $q \neq 0, 1$, we have a long exact sequence

$$\cdots \longrightarrow E_p \longrightarrow E_{p,0}^2 \xrightarrow{d_{p,0}^2} E_{p-2,1}^2 \longrightarrow E_{p-1} \longrightarrow \cdots.$$

Hence, if $p-1 > q_0$, we have $E_p = E_{p-1} = 0$ and the map $d_{p,0}^2 : E_{p,0}^2 \to E_{p-2,1}^2$ is an isomorphism.

Proof of Theorem 2.10. 1. The assertion is local on X. Shrinking X if necessary, we take a regular immersion $X \to P$ of codimension 1 into a smooth scheme P over S. We verify the assumption $\mathcal{T}or_q^{O_P}(\mathcal{F},\mathcal{G}) = 0$ for $q > q_0 = b+n+m$ of Lemma 2.29 is satisfied. We consider the spectral sequence $E_{p,q}^2 = \mathcal{T}or_p^{O_{P\times_S P}}(O_P, \mathcal{T}or_q^{O_S}(\mathcal{F},\mathcal{G})) \Rightarrow \mathcal{T}or_{p+q}^{O_P}(\mathcal{F},\mathcal{G})$. By the assumption, the E^2-term vanishes for $q > b+m$. Further by the assumption, the smooth scheme P is of relative dimension n. Hence the diagonal map $P \to P\times_S P$ is a regular immersion of codimension n and the conormal sheaf $N_{P/P\times_S P} = \Omega^1_{P/S}$ is locally free of rank n. By Lemma 2.26, we have $E_{p,q}^2 = \Omega^p_{P/S} \otimes_{O_P} \mathcal{T}or_q^{O_S}(\mathcal{F},\mathcal{G})$. Hence the E^2-term vanishes also for $p > n$. Therefore, we have $\mathcal{T}or_q^{O_P}(\mathcal{F},\mathcal{G}) = 0$ for $q > q_0$.

Applying Lemma 2.29, we see that the map $\iota_{\mathcal{F},\mathcal{G},X/P} : \mathcal{T}or_q^{O_X}(\mathcal{F},\mathcal{G}) \to \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P}$ is an isomorphism for $q-2 \geq q_0$. By Corollary 2.28, the composite

$$\mathcal{T}or_q^{O_X}(\mathcal{F},\mathcal{G}) \longrightarrow \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P} \xrightarrow{\mathrm{id}\otimes d} \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes \Omega^1_{P/S}|_X$$

is the 0-map. Therefore the map $\mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P} \to \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes \Omega^1_{P/S}|_X$ is the 0-map. By the description of the closed subscheme Z in the proof of Lemma 2.9, it means that $\mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G})$ is an $O_{Z_{W_V}}$-module for $q-2 \geq q_0$.

2. Let U be an open subscheme of X and $U \to P$ be a regular immersion. Then by Corollary 2.28, we have a map

$$\mathcal{T}or_q^{O_X}(\mathcal{F},\mathcal{G})|_U \to \mathrm{Ker}(\mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G})|_U \otimes N_{U/P} \to \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G})|_U \otimes \Omega^1_{P/S}|_U).$$

By 1, the right hand side is $(\mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes_{O_Z} \mathcal{L}_Z)|_{Z_{W_V}\times_X U}$ for $q-2 \geq q_0$. By Lemma 2.27, the maps are glued to define the map $\iota_{\mathcal{F},\mathcal{G},X/S} : \mathcal{T}or_q^{O_X}(\mathcal{F},\mathcal{G}) \to \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes \mathcal{L}_Z$. If the codimension of the immersion $U \to P$ is 1, it is an isomorphism by Lemma 2.29. Since X is locally a hypersurface, the map is an isomorphism.

The functoriality is clear from the definition.

Proof of Lemma 2.12. 1. The proofs are similar and we show the first equality $[[\mathcal{F},\mathcal{G}]] = [[\mathcal{F}',\mathcal{G}]] + [[\mathcal{F}'',\mathcal{G}]]$ for an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. By the long exact sequence

$$\begin{array}{llll}\longrightarrow \mathcal{T}or_q^{O_X}(\mathcal{F}',\mathcal{G}) & \xrightarrow{a} \mathcal{T}or_q^{O_X}(\mathcal{F},\mathcal{G}) & \longrightarrow \mathcal{T}or_q^{O_X}(\mathcal{F}'',\mathcal{G}) \\ \longrightarrow \mathcal{T}or_{q-1}^{O_X}(\mathcal{F}',\mathcal{G}) & \longrightarrow \mathcal{T}or_{q-1}^{O_X}(\mathcal{F},\mathcal{G}) & \longrightarrow \mathcal{T}or_{q-1}^{O_X}(\mathcal{F}'',\mathcal{G}) \\ \longrightarrow \mathcal{T}or_{q-2}^{O_X}(\mathcal{F}',\mathcal{G}) & \xrightarrow{b} \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) & \longrightarrow & ,\end{array}$$

it is sufficient to show that Im a is isomorphic to Im $b \otimes \mathcal{L}_Z$. It follows from the functoriality of the canonical isomorphism.

2. Assume $F^q\mathcal{G}$ is acyclic for $q \leq a$ and $\mathcal{G}/F^q\mathcal{G}$ is acyclic for $q \geq b$. By induction on $b-a$ it is reduced to the case where $a=-1$ and $b=1$. In other words, it is sufficient to show an equality $[[\mathcal{F},\mathcal{G}]] = [[\mathcal{F},\mathcal{G}']] + [[\mathcal{F},\mathcal{G}'']]$ for an exact sequence $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ of complexes. It is proved similarly as in 1.

Proof of Proposition 2.14. The canonical quasi-isomorphism $L_{X/S} \to L_{X/S,P}$ induces an isomorphism $\mathcal{L}_Z \to N_{X/P}|_Z$. Hence the canonical map $G(Z_{W_V}) \to G(W_V)$ induces a map $G(Z_{W_V})_{/\mathcal{L}_Z} \to G(W_V)_{/N_{X/P}}$.

It is enough to show the equality $(\mathcal{F},\mathcal{G})_P = [[\mathcal{F},\mathcal{G}]]_X$ in $G(W_V)_{/N_{X/P}}$ for a coherent O_V-module $\mathcal{F}$ and a coherent O_W-module $\mathcal{G}$. We consider the spectral sequence

$$E^2_{p,q} = \mathcal{T}or_p^{O_X}(\mathcal{T}or_q^{O_P}(\mathcal{F},O_X),\mathcal{G}) \Rightarrow \mathcal{T}or_{p+q}^{O_X}(\mathcal{F},\mathcal{G})$$

as in the proof of Lemma 2.29. Then it is shown there that we have a long exact sequence

$$\longrightarrow \mathcal{T}or_p^{O_P}(\mathcal{F},\mathcal{G}) \longrightarrow \mathcal{T}or_p^{O_X}(\mathcal{F},\mathcal{G}) \longrightarrow \mathcal{T}or_{p-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P} \longrightarrow .$$

Assume $\mathcal{T}or_p^{O_P}(\mathcal{F},\mathcal{G}) = 0$ for $p > m$. Then, in $G(W_V)_{/N_{X/P}}$, we have

$$\begin{aligned}(\mathcal{F},\mathcal{G})_P =& \sum_{p=0}^{m}(-1)^p[\mathcal{T}or_p^{O_P}(\mathcal{F},\mathcal{G})]\\ =& \sum_{p=0}^{m+1}(-1)^p[\mathcal{T}or_p^{O_X}(\mathcal{F},\mathcal{G})] - \sum_{p=0}^{m-1}(-1)^p[\mathcal{T}or_p^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P}]\\ =&(-1)^m[\mathcal{T}or_m^{O_X}(\mathcal{F},\mathcal{G})] + (-1)^{m+1}[\mathcal{T}or_{m+1}^{O_X}(\mathcal{F},\mathcal{G}) \otimes N_{X/P}] = [[\mathcal{F},\mathcal{G}]]_X.\end{aligned}$$

Proof of Propositions 2.19 and 2.23 are similar and we give the proof of Proposition 2.23 first.

Proof of Proposition 2.23. 1. We consider the spectral sequence

$$E^2_{p,q} = \mathcal{T}or_p^{O_W}(\mathcal{T}or_q^{O_X}(\mathcal{F},\mathcal{G}),\mathcal{H}) \Rightarrow \mathcal{T}or_{p+q}^{O_X}(\mathcal{F},\mathcal{G} \otimes^L_{O_W} \mathcal{H}).$$

Let p_0 be the tor-dimension of the complex $\mathcal{H}$ of O_W-modules so that we have $E^2_{p,q} = 0$ for $p > p_0$. Let $m-2 \geq p_0+q_0$ be a sufficiently large integer. Then the left hand side $[[\mathcal{F},\mathcal{G}\otimes_{O_W}\mathcal{H}]]_X$ is equal to $(-1)^m(\sum_{p+q=m}[E^\infty_{p,q}] - \sum_{p+q=m+1}[E^\infty_{p,q}])$. On the other hand, for an integer integer $q-2 \geq q_0$, we have $([\mathcal{T}or_q^{O_X}(\mathcal{F},\mathcal{G})],\mathcal{H})_W = \sum_p(-1)^p[E^2_{p,q}]$. Hence the right hand side $([[\mathcal{F},\mathcal{G}]]_X,\mathcal{H})_W$ is $\sum_p\sum_{q=q_1}^{q_1+1}(-1)^{p+q}[E^2_{p,q}]$ for an integer $q_1-2 \geq q_0$. For $q-2 \geq q_0$, we have $[E^2_{p,q}] = [E^2_{p,q-2}]$ by the isomorphism $\mathcal{T}or_q^{O_X}(\mathcal{F},\mathcal{G}) \to \mathcal{T}or_{q-2}^{O_X}(\mathcal{F},\mathcal{G}) \otimes \mathcal{L}_Z$. Hence, we have $\sum_p\sum_{q=q_1}^{q_1+1}(-1)^{p+q}[E^2_{p,q}] = \sum_{p+q=m,m+1}(-1)^{p+q}[E^2_{p,q}]$. Therefore, it is sufficient to prove an equality

$$\sum_{p+q=m,m+1}(-1)^{p+q}[E^2_{p,q}] = \sum_{p+q=m,m+1}(-1)^{p+q}[E^\infty_{p,q}]$$

for $m-2 \geq p_0+q_0$, to complete the proof. It is reduced to an equality

$$\sum_{p+q=m,m+1} (-1)^{p+q}[E^r_{p,q}] = \sum_{p+q=m,m+1} (-1)^{p+q}[E^{r+1}_{p,q}]$$

for $r \geq 2$. The difference is equal to

$$\sum_{p+q=m,m+1} (-1)^{p+q}([\mathrm{Im}\ d^r_{p,q}] + [\mathrm{Im}\ d^r_{p+r,q-r+1}]) = (-1)^m(\sum_{p+q=m} [\mathrm{Im}\ d^r_{p,q}] - \sum_{p+q=m-2} [\mathrm{Im}\ d^r_{p,q}]).$$

It is reduced to the following Lemma since, for $q-2 \geq q_0$, the inequality $p+q-2 \geq p_0+q_0$ implies $(r-1)(q-2-q_0) \geq (r-2)(p_0-p)$ for $r \geq 2$.

Lemma 2.31 *The isomorphism $\iota_q : E^2_{p,q} \to E^2_{p,q-2} \otimes \mathcal{L}_Z$ induces isomorphisms $E^r_{p,q} \to E^r_{p,q-2} \otimes \mathcal{L}_Z$ and $\mathrm{Im}\ d^r_{p,q} \to \mathrm{Im}\ d^r_{p,q-2} \otimes \mathcal{L}_Z$ for $(r-1)(q-2-q_0) \geq (r-2)(p_0-p)$.*

Proof of Lemma 2.31. We prove it by induction on r. The assertion is local on X. Shrinking X if necessary, we take a regular immersion $X \to P$ of codimension 1 into a smooth scheme P over S. Then the map ι_q is induced by the map $\iota : \mathcal{F} \to \mathcal{F} \otimes N_{X/P}[2]$. We also consider the spectral sequence

$$E'^2_{p,q} = \mathcal{T}or^{O_W}_p(\mathcal{T}or^{O_X}_{q-2}(\mathcal{F},\mathcal{G}),\mathcal{H}) \otimes N_{X/P} \Rightarrow \mathcal{T}or^{O_X}_{p+q-2}(\mathcal{F},\mathcal{G} \otimes^L_{O_W} \mathcal{H}) \otimes N_{X/P}.$$

The the map ι induces a map of spectral sequences $E^2_{p,q} \to E'^2_{p,q} = E^2_{p,q-2} \otimes \mathcal{L}_Z$ and the map on the E^2-terms is the same as $\iota_q : E^2_{p,q} \to E^2_{p,q-2} \otimes \mathcal{L}_Z$. We consider the commutative diagram

$$\begin{array}{ccccc}
E^r_{p+r,q-r+1} & \xrightarrow{d^r_{p+r,q-r+1}} & E^r_{p,q} & \xrightarrow{d^r_{p,q}} & E^r_{p-r,q+r-1} \\
\downarrow & & \downarrow & & \downarrow \\
E'^r_{p+r,q-r+1} & \xrightarrow[d^r_{p+r,q-r+1}]{} & E'^r_{p,q} & \xrightarrow[d^r_{p,q}]{} & E'^r_{p-r,q+r-1}.
\end{array}$$

We show that the vertical maps are isomorphisms for $(r-1)(q-2-q_0) \geq (r-2)(p_0-p)$ by induction on $r \geq 2$. If $r=2$, we have $q-2 \geq q_0$ and hence isomorphisms. Since $(r-1)(q-2-q_0) \geq (r-2)(p_0-p)$ and $p \leq p_0$ implies $(r-2)(q-2-q_0) \geq (r-3)(p_0-p)$ for $r \geq 3$, the assertion follows by induction on r.

2. We show that the both sides are equal to $[\mathcal{H}_q(\mathcal{F} \otimes^L_{O_X} \mathcal{G} \otimes^L_{O_{X'}} \mathcal{F}')] - [\mathcal{H}_{q+1}(\mathcal{F} \otimes^L_{O_X} \mathcal{G} \otimes^L_{O_{X'}} \mathcal{F}')]$ for sufficiently large even q. We apply Lemma 2.12.2 by taking $\mathcal{G} \otimes^L_{O_{X'}} \mathcal{F}'$ to be $\mathcal{G}$ in Lemma 2.12.2 and F to be the canonical filtration. Then we obtain

$$[[\mathcal{F}, \mathcal{G} \otimes^L_{O_{X'}} \mathcal{F}']] = \sum_q (-1)^q [[\mathcal{F}, \mathcal{T}or^{O_{X'}}_q(\mathcal{G},\mathcal{F}')]].$$

The right hand side is equal to $[[\mathcal{F}, (\mathcal{F}',\mathcal{G})_{X'}]]_X$ and the left hand side is equal to $[\mathcal{H}_q(\mathcal{F} \otimes^L_{O_X} \mathcal{G} \otimes^L_{O_{X'}} \mathcal{F}')] - [\mathcal{H}_{q+1}(\mathcal{F} \otimes^L_{O_X} \mathcal{G} \otimes^L_{O_{X'}} \mathcal{F}')]$ for sufficiently large even q. The equality for $[[\mathcal{F}', (\mathcal{F},\mathcal{G})_X]]_{X'}$ is proved in the same way.

Proof of Proposition 2.19. 1. By the projection formula $R\pi_*(L\pi^*\mathcal{F} \otimes^L_{O_{X'}} \mathcal{G}) = \mathcal{F} \otimes^L_{O_X} R\pi_*\mathcal{G}$, we have a spectral sequence

$$E^2_{p,q} = R^{-p}\pi_*\mathcal{T}or_q^{O_{X'}}(L\pi^*\mathcal{F}, \mathcal{G}) \Rightarrow \mathcal{T}or_{p+q}^{O_X}(\mathcal{F}, R\pi_*\mathcal{G}).$$

Similarly as in the proof of Proposition 2.23.2, we have an equality

$$[[\mathcal{F}, R\pi_*\mathcal{G}]]_X = \sum_{p+q=m,m+1} (-1)^{p+q}[E^\infty_{p,q}]$$

for a sufficiently large integer m, by Lemma 2.12.2. On the other hand, by the isomorphism $\mathcal{T}or_q^{O_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{T}or_q^{O_{X'}}(L\pi^*\mathcal{F}, \mathcal{G})$, we have

$$\pi_*[[\mathcal{F}, \mathcal{G}]]_{X'} = \sum_p \sum_{q=m,m+1} (-1)^{p+q}[E^2_{p,q}].$$

The rest of proof is parallel to that of Proposition 2.23 and left to the reader.

The proof of 2 is similar and is also left to the reader.

2.4. Review on derived exterior power complex.

We recall the derived exterior power complex to prove Proposition 2.16 in the next subsection. Basic references are [31], [30] and [14]. In this paper, we use the second definition of the derived exterior power complex in [31].

Definition 2.32 ([31] Section 4 Definitions 1 and 2, Lemma 4.2) *1. For a quasi-coherent O_X-module $\mathcal{F}$ and an integer n, the n-th symmetric tensor module $TS^n\mathcal{F}$ is the fixed part*

$$TS^n\mathcal{F} = \bigcap_{\sigma \in \mathcal{S}_n} \mathrm{Ker}(\sigma - 1 : \mathcal{F}^{\otimes n} \to \mathcal{F}^{\otimes n})$$

by the natural action of the symmetric group $\mathcal{S}_n$. The inclusion defines a canonical map $TS^{p+q}\mathcal{F} \to TS^p\mathcal{F} \otimes TS^q\mathcal{F}$.

2. Let $f : \mathcal{L} \to \mathcal{E}$ be a morphism of locally free O_X-modules of finite ranks and let $\mathcal{K} = [\mathcal{L} \xrightarrow{f} \mathcal{E}]$ be the complex of length 1 where $\mathcal{E}$ is put on degree 0. For an integer $q, n \geq 0$, we put $(L\Lambda^q\mathcal{K})_n = \Lambda^{q-n}\mathcal{E} \otimes TS^n\mathcal{L}$. We define a map $d_n : \Lambda^{q-n}\mathcal{E} \otimes TS^n\mathcal{L} \to \Lambda^{q-n+1}\mathcal{E} \otimes TS^{n-1}\mathcal{L}$ to be the composition

$$\Lambda^{q-n}\mathcal{E} \otimes TS^n\mathcal{L} \xrightarrow{\mathrm{id}\otimes\mathrm{can}} \Lambda^{q-n}\mathcal{E} \otimes \mathcal{L} \otimes TS^{n-1}\mathcal{L} \xrightarrow{\mathrm{id}\otimes f\otimes\mathrm{id}}$$
$$\Lambda^{q-n}\mathcal{E} \otimes \mathcal{E} \otimes TS^{n-1}\mathcal{L} \xrightarrow{\wedge\otimes\mathrm{id}} \Lambda^{q-n+1}\mathcal{E} \otimes TS^{n-1}\mathcal{L}.$$

We define a complex $L\Lambda^q\mathcal{K}$ to be

$$\begin{array}{ccccccc} [TS^q\mathcal{L} & \xrightarrow{d_q} & \mathcal{E} \otimes TS^{q-1}\mathcal{L} & \longrightarrow & \cdots & \longrightarrow & \\ \Lambda^{q-n}\mathcal{E} \otimes TS^n\mathcal{L} & \xrightarrow{d_n} & \cdots & \longrightarrow & \Lambda^{q-1}\mathcal{E} \otimes \mathcal{L} & \xrightarrow{d_1} & \Lambda^q\mathcal{E}]. \end{array}$$

The last term $\Lambda^q\mathcal{E}$ is put on degree 0. For $q < 0$, we put $L\Lambda^q\mathcal{K} = 0$.

If $\mathcal{L}$ is invertible, we have $L\Lambda^q\mathcal{K} = [L^{\otimes q} \to \mathcal{E} \otimes \mathcal{L}^{\otimes q-1} \to \cdots \to \Lambda^q\mathcal{E}]$.

Lemma 2.33 ([31] Corollary (2) of Lemma 4.1) *If $f : \mathcal{K} \to \mathcal{K}'$ is a quasi-isomorphism, the induced map $L\Lambda^q\mathcal{K} \to L\Lambda^q\mathcal{K}'$ is a quasi-isomorphism.*

Let $\mathcal{F}$ be a coherent O_X-module and assume there exists a locally free resolution $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{F} \to 0$ of length 1. Then the complex $L\Lambda^q\mathcal{F}$ is defined as $L\Lambda^q\mathcal{K}$ where $\mathcal{K}$ is the complex $[\mathcal{L} \to \mathcal{E}]$. It is well-defined in the derived category by Lemma 2.33. We have a canonical isomorphism $L^0\Lambda^q\mathcal{K} = \mathcal{H}_0(L\Lambda^q\mathcal{K}) \to \Lambda^q\mathcal{F}$.

The complexes $L\Lambda^{q+1}\mathcal{K}$ and $L\Lambda^q\mathcal{K}$ are related as follows.

Lemma 2.34 *Let the notation be as in Lemma 1.7. Namely, let $f : \mathcal{L} \to \mathcal{E}$ be a morphism of locally free O_X-modules and $n \geq 0$ be an integer. Assume that $\mathcal{F} = \operatorname{Coker} f$ is locally generated by n sections and that $\operatorname{rank} \mathcal{E} - \operatorname{rank} \mathcal{L} = n - 1$. Let $\mathcal{K}$ be the complex $[\mathcal{L} \xrightarrow{f} \mathcal{E}]$ of length 1 where $\mathcal{E}$ is put on degree 0 and let $i : Z \to X$ be the closed immersion defined by the annihilater ideal $\operatorname{Ann} \Lambda^n\mathcal{F}$. Then,*

1. The O_Z-module $\mathcal{L}_Z$ is invertible. For $p > 0$, the homology sheaf $L^p\Lambda^q\mathcal{K} = \mathcal{H}_p(L\Lambda^q\mathcal{K})$ is an O_Z-module.

2. There is a canonical isomorphism $\iota_{\mathcal{K}} : L^{p+1}\Lambda^{q+1}\mathcal{K} \to L^p\Lambda^q\mathcal{K} \otimes_{O_Z} \mathcal{L}_Z$ for $p > 0$.

Proof. 1. The assertion for $\mathcal{L}_Z$ is shown in Lemma 1.7. Shrinking X and replacing the complex $\mathcal{K}$ by a complex quasi-isomorphic to it, we may assume that $\mathcal{E} = O_X^n, \mathcal{L} = O_X$ and the map $\mathcal{L} \to \mathcal{E}$ is defined by a vector $a = (a_1, \ldots, a_n) \in O_X^n$. The exterior power complex $L\Lambda^q\mathcal{K}$ is then given by $[\mathcal{L}^{\otimes q} \to \mathcal{L}^{\otimes q-1} \otimes \mathcal{E} \to \cdots \to \Lambda^q\mathcal{E}]$. It is isomorphic to a part of Koszul complex $\operatorname{Kos}(O_X^n \xrightarrow{a} O_X)$ defined by $a = (a_1, \ldots, a_n)$. Hence the cohomology sheaf $L^p\Lambda^q\mathcal{K}$ is annihilated by the ideal $I_Z = (a_1, \ldots, a_n)$ for $p > 0$.

2. We define a map $L\Lambda^{q+1}\mathcal{K}[1] \to L\Lambda^q\mathcal{K} \otimes \mathcal{L}$ of complexes by the canonical maps $\Lambda^{q-p}\mathcal{E} \otimes TS^{p+1}\mathcal{L} \to \Lambda^{q-p}\mathcal{E} \otimes TS^p\mathcal{L} \otimes \mathcal{L}$. By 1, it induces a map $L^{p+1}\Lambda^{q+1}\mathcal{K} \to L^p\Lambda^q\mathcal{K} \otimes \mathcal{L}|_Z$. We show that it induces the required isomorphism $L^{p+1}\Lambda^{q+1}\mathcal{K} \to L^p\Lambda^q\mathcal{K} \otimes \mathcal{L}_Z$. The question is local and, by functoriality of the definition above, we may replace $\mathcal{K}$ by a complex quasi-isomorphic to it. Hence we may assume $\mathcal{L}$ is invertible. Then $\mathcal{L}_Z = \mathcal{L}|_Z$ and it is clear.

Let X be a noetherian scheme. Under the assumption of Lemma 2.34, the complex $L\Lambda^n\mathcal{K}$ is acyclic outside Z. Hence, we have a map $(\ ,[L\Lambda^n\mathcal{K}])_X : G(X) \to G(Z)$ sending the class $[\mathcal{F}]$ of a coherent O_X-module $\mathcal{F}$ to $[\mathcal{F} \otimes_{O_X} L\Lambda^n\mathcal{K}]$. It is related to the localized chern class $c_{n}{}^{X}_{Z}(\mathcal{K})$ in the following way.

Proposition 2.35 *Let the notation be as in Lemma 1.7. Namely, let $f : \mathcal{L} \to \mathcal{E}$ be a morphism of locally free O_X-modules and $n \geq 0$ be an integer. Assume that $\mathcal{F} = \operatorname{Coker} f$ is locally generated by n sections and that $\operatorname{rank} \mathcal{E} - \operatorname{rank} \mathcal{L} = n - 1$. Let $\mathcal{K}$ be the complex $[\mathcal{L} \xrightarrow{f} \mathcal{E}]$ of length 1 where $\mathcal{E}$ is put on degree 0 and let $i : Z \to X$ be the closed immersion defined by the annihilater ideal $\operatorname{Ann} \Lambda^n\mathcal{F}$.*

Assume X is of finite type over a regular noetherian scheme. Then the map $(\ ,[L\Lambda^n\mathcal{K}])_X : G(X)\to G(Z)$ sends the topological filtration $F_mG(X)$ to $F_{m-n}G(Z)$. For the induced map, we have a commutative diagram

$$\begin{array}{ccc} CH_m(X) & \xrightarrow{c_n{}_Z^X(\mathcal{K})\cap} & CH_{m-n}(Z) \\ \downarrow & & \downarrow \\ Gr_m^F G(X) & \xrightarrow[(\ ,[L\Lambda^n\mathcal{K}])_X]{} & Gr_{m-n}^F G(Z) \end{array}$$

We reduce it to the following computation.

Lemma 2.36 *Let the notation be as in Proposition 2.35. Let $\varphi : W \to X$ be a morphism of noetherian schemes.*

1. Assume W is a scheme over Z. Then $\mathcal{E}_W = L^0\varphi^*\mathcal{K}$ is a locally free O_W-modules of rank n and $\mathcal{L}_W = L^1\varphi^*\mathcal{K}$ is an invertible O_W-modules as in Lemma 1.7.2. Further we have

$$\begin{aligned}[L\varphi^*L\Lambda^n\mathcal{K}] =&(-1)^n\sum_p(-1)^p[\Lambda^p(\mathcal{E}_W\otimes\mathcal{L}_W^{\otimes-1})\otimes\mathcal{L}_W^{\otimes n}]\\ =&\gamma_n([\mathcal{E}_W\otimes\mathcal{L}_W^{\otimes-1}]-n)[\mathcal{L}_W]^n\end{aligned}$$

in $K(W)$.

2. Assume $D = Z\times_X W$ is a divisor of W. Let $\varphi_D : D\to Z$ be the restriction. We put $\mathcal{L}_D = L^1(i\circ\varphi_D)^*\mathcal{K} = \varphi_D^*\mathcal{L}_Z$ and let $\mathcal{E}'_W$ be the locally free quotient of $\varphi^*\mathcal{F}$ by $\mathcal{L}_D\otimes_{O_W}O_W(D)$ as in Lemma 1.7.3. Then the complex $L\varphi^*L\Lambda^n\mathcal{K}$ of O_W-modules is acyclic outside D and its class in $G(D)$ is given by

$$\begin{aligned}[L\varphi^*L\Lambda^n\mathcal{K}] =&\ (-1)^{n-1}\sum_p(-1)^p[\Lambda^p(\mathcal{E}'_W|_D\otimes\mathcal{L}_D^{\otimes-1})\otimes\mathcal{L}_D^{\otimes n}(D)|_D]\\ =&\ \gamma_{n-1}([\mathcal{E}'_W|_D\otimes\mathcal{L}_D^{\otimes-1}]-(n-1))[\mathcal{L}_D]^n[O_D(D)].\end{aligned}$$

Proof of Lemma. 1. In this case, we have an isomorphism $L^q\varphi^*L\Lambda^n\mathcal{K}\to\Lambda^{n-q}\mathcal{E}_W\otimes\mathcal{L}_W^{\otimes q}$. Hence we have

$$[L\varphi^*L\Lambda^n\mathcal{K}] = \sum_p(-1)^{n-p}[\Lambda^p\mathcal{E}_W\otimes\mathcal{L}_W^{\otimes n-p}] = (-1)^n\sum_p(-1)^p[\Lambda^p(\mathcal{E}_W\otimes\mathcal{L}_W^{\otimes-1})\otimes\mathcal{L}_W^{\otimes n}].$$

The second equality in the statement follows from Lemma 2.5.

2. We show that $L^q\varphi^*L\Lambda^n\mathcal{K}$ is isomorphic to $\Lambda^{n-q-1}\mathcal{E}'_W|_D\otimes\mathcal{L}_D^{\otimes q+1}(D)|_D$. Once this is done, we have

$$\begin{aligned}[L\varphi^*L\Lambda^n\mathcal{K}] =&\sum_p(-1)^{n-1-p}[\Lambda^p\mathcal{E}'_W|_D\otimes_{O_D}\mathcal{L}_D^{\otimes n-p}(D)|_D]\\ =&(-1)^{n-1}\sum_p(-1)^p[\Lambda^p(\mathcal{E}'_W|_D\otimes\mathcal{L}_D^{\otimes-1})\otimes_{O_D}\mathcal{L}_D^{\otimes n}(D)|_D].\end{aligned}$$

The second equality in the statement follows from Lemma 2.5. We show the isomorphism. As in the proof of Lemma 1.7.2, we may assume $X = W$ and $Z = D$ is a divisor of X. We define an isomorphism locally on X. Let U be an open subscheme of X and $[\mathcal{L}_U \to \mathcal{E}_U] \to \mathcal{K}|_U$ be an quasi-isomorphism where $\mathcal{E}_U$ is a locally free O_U-module of rank n and $\mathcal{L}_U$ is an invertible O_U-module. The kernel of the composition $\mathcal{E}_U \to \mathcal{F}|_U \to \mathcal{E}'|_U$ is identified with $\mathcal{L}'_U = \mathcal{L}_U(D)$ and we have an exact sequence $0 \to \mathcal{L}'_U \to \mathcal{E}_U \to \mathcal{E}'|_U \to 0$. We have a canonical isomorphism $\mathcal{L}_U|_{D\cap U} = \mathcal{L}_D|_{D\cap U}$. Taking the exterior power, we obtain an exact sequence $0 \to \Lambda^{n-q-1}\mathcal{E}'|_U \otimes \mathcal{L}'_U \to \Lambda^{n-q}\mathcal{E}_U \to \Lambda^{n-q}\mathcal{E}'|_U \to 0$. From this and the definition of the complex $L\Lambda^n\mathcal{K}$, we deduce easily an isomorphism

$$\begin{aligned} L^q\Lambda^n\mathcal{K}|_U &= \mathrm{Ker}(\Lambda^{n-q}\mathcal{E}_U \otimes \mathcal{L}_U^{\otimes q} \to \Lambda^{n-q+1}\mathcal{E}_U \otimes \mathcal{L}_U^{\otimes q-1})/\mathrm{Im}(\Lambda^{n-q-1}\mathcal{E}_U \otimes \mathcal{L}_U^{\otimes q+1} \to \Lambda^{n-q}\mathcal{E}_U \otimes \mathcal{L}_U^{\otimes q}) \\ &\to (\Lambda^{n-q-1}\mathcal{E}'|_U \otimes \mathcal{L}'_U \otimes \mathcal{L}_U^{\otimes q})/(\Lambda^{n-q-1}\mathcal{E}'|_U \otimes \mathcal{L}_U^{\otimes q+1}) = (\Lambda^{n-q-1}\mathcal{E}'|_D \otimes \mathcal{L}_D^{\otimes q+1}(D))|_{U\cap D}. \end{aligned}$$

It is easy to patch them together to obtain the required isomorphism.

Proof of Proposition 2.35. The topological filtration $F_pG(X)$ is generated by the classes $[O_W]$ for integral closed subschemes $W \subset X$ of dimension $\leq p$. If $W \subset Z$, we put $W' = W$ and if otherwise, let $\pi : W' \to W$ be the blow-up of W at $Z_W = Z \times_X W$. Then, the topological filtration $F_pG(X)$ is generated by the classes $\pi_*[O_{W'}] = [R\pi_*O_{W'}]$ for integral closed subschemes $W \subset X$ of dimension $\leq p$. We have $\pi_*[L\Lambda^n L\pi^*\mathcal{K}] = [R\pi_*L\pi^*L\Lambda^n\mathcal{K}] = [L\Lambda^n\mathcal{K}] \cdot [R\pi_*O_X]$ by the spectral sequence $R^i\pi_*L^{-j}\pi^*L\Lambda\mathcal{K} \Rightarrow R^{i+j}\pi_*L\pi^*L\Lambda\mathcal{K}$ and by the projection formula $R\pi_*L\Lambda^n L\pi^*\mathcal{K} \simeq L\Lambda^n\mathcal{K} \otimes^L R\pi_*O_W$. Hence it is sufficient to show that $(W', [L\Lambda^n\mathcal{K}])_X$ is in $F_{p-n}G(Z_{W'})$ and its class in $Gr^F_{p-n}G(Z_{W'})$ is equal to the image of $c_n{}^X_Z(\mathcal{K}) \cap [W']$ assuming $\dim W = p$.

If $W \subset Z$, we have $(W, [L\Lambda^n\mathcal{K}])_X = \gamma_n([\mathcal{E}_W \otimes \mathcal{L}_W^{\otimes -1}] - n)[\mathcal{L}_W]^n$ by Lemma 2.36.1. Further by Lemma 2.5, it is contained in $F_{p-n}G(W)$ and its class in $Gr^F_{p-n}G(W)$ is equal to the image of $c_n(\mathcal{E}_W \otimes \mathcal{L}_W^{\otimes -1}) \cap [W]$. By Lemma 1.7.2, it is further equal to $c_n{}^X_Z(\mathcal{K}) \cap [W]$.

Assume $W \not\subset Z$ and $D = Z \times_X W'$ be the exceptional divisor. Then applying Lemma 2.36.2 to W', under the notation there, we have $(W', [L\Lambda^n\mathcal{K}])_X = \gamma_{n-1}([\mathcal{E}'_{W'} \otimes \mathcal{L}_D^{\otimes -1}] - n)[\mathcal{L}_D]^n[O_D(D)]$. Further by Lemma 2.5, it is contained in $F_{p-n}G(D)$ and its image in $Gr^F_{p-n}G(D)$ is equal to the image of $c_{n-1}(\mathcal{E}'_{W'}|_D \otimes \mathcal{L}_D^{\otimes -1}) \cap [D]$. By Lemma 1.7.3, it is further equal to $c_n{}^X_Z(\mathcal{K}) \cap [W']$. Thus proof is completed.

We recall a generalization of Koszul complex. It will be used to prove the excess intersection formula, Proposition 2.16, in the following subsection §2.5. First we recall the usual Koszul complex. Let $a : \mathcal{E} \to O_X$ be a morphism of O_X-modules. Then the Koszul complex $\mathrm{Kos}(\mathcal{E} \xrightarrow{a} O)$ is defined to be the complex $[\to \Lambda^n\mathcal{E} \xrightarrow{d_n} \Lambda^{n-1}\mathcal{E} \to \cdots \to \mathcal{E} \to O_X]$ where O_X is put on degree 0 and

$$d_n(e_1 \wedge \cdots \wedge e_n) = \sum_{i=1}^n (-1)^{i-1} a(e_i) e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_n.$$

Let $f : \mathcal{L} \to \mathcal{E}$ be a morphism of O_X-modules satisfying $a \circ f = 0$. Let $\mathcal{K} = [\mathcal{L} \xrightarrow{f} \mathcal{E}]$ be a complex of length 1 and regard a as a morphism $a : \mathcal{K} \to O_X$ of complexes. The generalized Koszul complex $\mathcal{G} = \mathrm{Kos}(\mathcal{K} \xrightarrow{a} O)$ is the double complex $(\Lambda^{q-p}\mathcal{E} \otimes TS^p\mathcal{L}, d_{p,q}, d'_{p,q})$ defined as follows. The first partial complex $(\Lambda^{q-\bullet}\mathcal{E} \otimes TS^\bullet\mathcal{L}, d'_{\bullet,q})$ is the same as the q-th derived exterior power $L\Lambda^q\mathcal{K}$ and the second partial complex $(\Lambda^{\bullet-p}\mathcal{E}\otimes TS^p\mathcal{L}, d''_{p,\bullet})$ is the same as $\mathrm{Kos}(\mathcal{E}\xrightarrow{a} O)\otimes TS^p\mathcal{L}[p]$. It is easily verified that this defines a double complex.

We consider an elementary filtration $F_p\mathcal{G}$ of the double complex $\mathcal{G}$ defined by $F_p\mathcal{G}_{p',q}=\mathcal{G}_{p',q}$ if $p'\leq p$ and is 0 otherwise. We consider the simple complex $s\mathcal{G}$ as a filtered complex by the induced filtration. The graded complex $Gr^F_p\mathrm{Kos}(\mathcal{K}\xrightarrow{a} O_X)$ is the same as $\mathrm{Kos}(\mathcal{E}\xrightarrow{a} O_X)\otimes TS^p\mathcal{L}[p]$. The filtration defines a spectral sequence $E^1_{p,q}=\mathcal{H}_{q-p}(\mathrm{Kos}(\mathcal{E}\xrightarrow{a} O_X))\otimes TS^p\mathcal{L}\Rightarrow\mathcal{H}_{p+q}s\mathcal{G}$.

The map $\mathrm{id}\otimes\mathrm{can}:\mathcal{G}_{p.q}=\Lambda^{q-p}\mathcal{E}\otimes TS^p\mathcal{L}\to\mathcal{G}_{p-1,q-1}\otimes L\Lambda^{q-p}\mathcal{E}\otimes TS^{p-1}\mathcal{L}\otimes\mathcal{L}$ induces a canonical map $\mathcal{G}\to\mathcal{G}\otimes\mathcal{L}[1,1]$ of double complexes. It induces a map $s\mathcal{G}\to s\mathcal{G}\otimes\mathcal{L}[2]$ of filtered complexes. If $\mathcal{L}$ is invertible, the simple complex $s\mathcal{G}$ is naturally identified with the cone of the map $s\mathcal{G}\otimes\mathcal{L}[1]\to\mathrm{Kos}(\mathcal{E}\to O)$. Then the map $s\mathcal{G}\to s\mathcal{G}\otimes\mathcal{L}[2]$ is the same as the canonical map $\mathrm{Cone}(s\mathcal{G}\otimes\mathcal{L}[1]\to\mathrm{Kos}(\mathcal{E}\to O))\to s\mathcal{G}\otimes\mathcal{L}[2]$.

We study functoriality of generalized Koszul complexes. To state it, we use the following Lemma.

Lemma 2.37 *For a scheme X, a locally free O_X-module $\mathcal{E}$ of finite rank and an integer p, there exists a canonical map $t_p: TS^p(\Lambda^2\mathcal{E})\to\Lambda^{2p}\mathcal{E}$ characterized by the following properties.*

(1) It commutes with an arbitrary base change.

(2) The composition with the map $S^p(\Lambda^2\mathcal{E})\to TS^p(\Lambda^2\mathcal{E}): x_1\cdots x_p\mapsto\sum_{\sigma\in\mathcal{S}_p}\sigma(x_1\otimes\cdots\otimes x_p)$ maps $x_1\cdots x_p$ to $x_1\wedge\cdots\wedge x_p$.

Proof. First we consider the case where X is flat over $\mathbf{Z}$. Let $s_p: TS^p(\Lambda^2\mathcal{E})\to\Lambda^{2p}\mathcal{E}$ be the composition of the canonical map $TS^p(\Lambda^2\mathcal{E})\to S^p(\Lambda^2\mathcal{E})$ with the map $S^p(\Lambda^2\mathcal{E})\to\Lambda^{2p}\mathcal{E}$ sending $x_1\cdot\cdots\cdot x_p$ to $x_1\wedge\cdots\wedge x_p$. It is sufficient to show that the map s_p is divisible by $p!$. Let $e_1,\ldots,e_n$ be a local basis of $\mathcal{E}$. For $k=((i_1,j_1),\ldots,(i_p,j_p))\in(\{1,\ldots,n\}^2)^p$, let $H_k\subset\mathcal{S}_p$ be the fixing subgroup of $e_{i_1}\wedge e_{j_1}\otimes\cdots\otimes e_{i_p}\wedge e_{j_p}$ and put $e_k=\sum_{\sigma\in\mathcal{S}_p/H_k}\sigma(e_{i_1}\wedge e_{j_1}\otimes\cdots\otimes e_{i_p}\wedge e_{j_p})$. The elements of the form e_k make a basis of $TS^p(\Lambda^2\mathcal{E})$. If $H_k\neq 1$, we have $s_p(e_k)=0$ and, if $H_k=1$, we have $s_p(e_k)=p!\cdot e_{i_1}\wedge e_{j_1}\otimes\cdots\otimes e_{i_p}\wedge e_{j_p}$. Thus the assertion is proved.

We prove the general case. We define the map locally on X by taking a basis of $\mathcal{E}$. Since $\mathcal{E}$ is locally a pull-back from Spec $\mathbf{Z}$, it is unique. We show the existence. Change of local bases defines a map to $GL_n=\mathrm{Spec}\,\mathbf{Z}[X_{11},\ldots,X_{nn},\det(X_{ij})^{-1}]$. Since GL_n is flat over $\mathbf{Z}$, the assertion follows.

Corollary 2.38 *Let $a:\mathcal{E}\to O_X$ be a map and $a_p:\Lambda^p\mathcal{E}\to\Lambda^{p-1}\mathcal{E}$ be the boundary map of the Koszul complex $\mathrm{Kos}(\mathcal{E}\xrightarrow{a} O_X)$. Then we have a commutative diagram*

$$\begin{array}{ccc}
TS^p(\Lambda^2\mathcal{E}) & \xrightarrow{t_p} & \Lambda^{2p}\mathcal{E}\\
\cap\downarrow & & \downarrow a_{2p-1}\\
\Lambda^2\mathcal{E}\otimes TS^{p-1}(\Lambda^2\mathcal{E}) & \xrightarrow[a_2\wedge t_{p-1}]{} & \Lambda^{2p-1}\mathcal{E}.
\end{array}$$

Proof. The assertion commutes with any base change. Hence we may assume $\mathcal{E}=O_X^n, X=\mathrm{Spec}\mathbf{Z}[T_1,\ldots,T_n]$ and the map a is defined by the vector $(T_1,\ldots,T_n)$. It is sufficient to show the commutativity on the image of the map $S^p(\Lambda^2\mathcal{E})\to TS^p(\Lambda^2\mathcal{E})$. By an elementary computation, we see that the image of $x_1\cdots x_p$ by the both maps are $\sum_{i=1}^p a_2(x_i)x_1\wedge\cdots\wedge x_{i-1}\wedge x_{i+1}\wedge x_p$.

Lemma 2.39 *Let* $\mathcal{K} = [\mathcal{L} \xrightarrow{f} \mathcal{E}] \xrightarrow{a} [0 \to O_X]$ *and* $\mathcal{K}' = [\mathcal{L}' \xrightarrow{f'} \mathcal{E}'] \xrightarrow{a'} [0 \to O_X]$ *be morphisms of complexes and let* $\mathcal{G} = \mathrm{Kos}(\mathcal{K} \xrightarrow{a} O)$ *and let* $\mathcal{G}' = \mathrm{Kos}(\mathcal{K}' \xrightarrow{a'} O)$ *be the generalized Koszul complexes. Let*

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{f} & \mathcal{E} & \xrightarrow{a} & O_X \\ {\scriptstyle (g_{\mathcal{L}},h)}\downarrow & & {\scriptstyle g_{\mathcal{E}}}\downarrow & & \| \\ \mathcal{L}' \oplus \Lambda^2\mathcal{E}' & \xrightarrow[(f',a'_2)]{} & \mathcal{E}' & \xrightarrow[a']{} & O_X \end{array}$$

be a commutative diagram of O_X*-modules. For* $0 \le r \le p \le q$*, we define* $g_{r,p,q} : \Lambda^{q-p}\mathcal{E} \otimes TS^p\mathcal{L} \to \Lambda^{q-p+2r}\mathcal{E}' \otimes TS^{p-r}\mathcal{L}'$ *to be* $(-1)^{qr+\binom{r+1}{2}}$*-times the restriction of the map* $\Lambda^{q-p}g_{\mathcal{E}} \wedge (t_{r,\mathcal{E}'} \circ TS^r h) \otimes TS^{p-r}g_{\mathcal{L}} : \Lambda^{q-p}\mathcal{E} \otimes TS^r\mathcal{L} \otimes TS^{p-r}\mathcal{L} \to \Lambda^{q-p+2r}\mathcal{E}' \otimes TS^{p-r}\mathcal{L}'$*. Then the maps* $(g_{p,q,r})_{p,q,r} : \bigoplus_{p+q=n} \Lambda^{q-p}\mathcal{E} \otimes TS^p\mathcal{L} \to \bigoplus_{p+q=n} \Lambda^{q-p}\mathcal{E}' \otimes TS^p\mathcal{L}'$ *define a map* $g : (s\mathcal{G}, F) \to (s\mathcal{G}', F)$ *of filtered simple complexes.*

Proof. It is sufficient to show the equality

$$g_{p-1,q,r-1} \circ d_{p,q} + (-1)^p g_{p,q-1,r} \circ d'_{p,q} = d_{p-r+1,q+r-1} \circ g_{p,q,r-1} + (-1)^{p-r} d'_{p-r,q+r} \circ g_{p,q,r}$$

of maps $\Lambda^{q-p}\mathcal{E} \otimes TS^p\mathcal{L} \to \Lambda^{q-p+2r-1}\mathcal{E}' \otimes TS^{p-r}\mathcal{L}'$ for each $p, q, r \ge 0$. Here we understand $g_{p,q,-1} = 0$. If $r \ge 2$, each term is the restriction of the map $\Lambda^{q-p}\mathcal{E} \otimes \mathcal{L} \otimes TS^{r-2}\mathcal{L} \otimes \mathcal{L} \otimes TS^{p-r}\mathcal{L} \to \Lambda^{q-p+2r-1}\mathcal{E}' \otimes TS^{p-r}\mathcal{L}'$ sending $e \otimes l_1 \otimes l_2 \otimes l_3 \otimes l_4$ to

$$\begin{gathered} (-1)^{q(r-1)+\binom{r}{2}} e' \wedge f(l_1)' \wedge t'_{r-1}(l_2 \otimes l_3) \otimes l'_4, \\ (-1)^{p+(q-1)r+\binom{r+1}{2}} a_{q-p}(e)' \wedge t'_r(l_1 \otimes l_2 \otimes l_3) \otimes l'_4, \\ (-1)^{q(r-1)+\binom{r}{2}} e' \wedge t'_{r-1}(l_1 \otimes l_2) \wedge f'(l'_3) \otimes l'_4, \\ (-1)^{p-r+qr+\binom{r+1}{2}} a'_{q-p+2r}(e' \wedge t'_r(l_1 \otimes l_2 \otimes l_3)) \otimes l'_4. \end{gathered}$$

Here an element with $'$ denotes the image by the maps induced by $g_{\mathcal{E}}$ or $g_{\mathcal{L}}$ of the elements without $'$ and t'_r denotes the composition $t_{r,\mathcal{E}'} \circ TS^r h$. By the definition of the Koszul complex and by Corollary 2.38, we have

$$a'_{q-p+2r}(e' \wedge t'_r(l_1 \otimes l_2 \otimes l_3)) = a'_{q-p}(e') \wedge t'_r(l_1 \otimes l_2 \otimes l_3) + (-1)^{q-p} e' \wedge a'_2(t'_1(l'_1)) \wedge t'_{r-1}(l_2 \otimes l_3).$$

We substitute this to the last term and the assumption $f(l_1)' = f'(l'_1) + a'_2(t'_1(l'_1))$ to the first term. Then the required equality is satisfied for the sections of $\Lambda^{q-p}\mathcal{E} \otimes TS^p\mathcal{L}$ by the invariance under $\mathcal{S}_p$. If $r = 0$ or 1, the proof is similar and easier.

2.5. Proof of excess intersection formula.

By Proposition 2.35 in the previous subsection, the excess intersection formula, Proposition 2.16, is reduced to the following more precise statement.

Proposition 2.40 *Let the assumption be as in Proposition 2.16. Namely, let* X *be locally a hypersurface of relative dimension* $n-1$ *over a noetherian scheme* S *and* $j : V \to X$ *be a closed* *subscheme of X. Assume that V is locally of complete intersection over S of relative dimension $n-c$ and that there exist a map $\mathcal{L} \to \mathcal{E}$ of locally free O_V-modules of finite rank and a quasi-isomorphism $[\mathcal{L} \to \mathcal{E}] \to M_{V/X}$. Let $\varphi : W \to X$ be a scheme over X. Assume that the immersion $T = Z \times_X W \to W$ is a regular immersion of codimension c' and let $M'_{V/X,W}$ be the excess conormal complex. Then we have*

$$[[V, W]]_X = (-1)^{c-c'}[L\Lambda^{c-c'} M'_{V/X,W}].$$

in $G(Z_T)_{/\mathcal{L}_Z}$. In particular, if $W = T$ is a scheme over V and $\psi : W \to V$ denotes the natural map, we have

$$[[V, W]]_X = (-1)^c[L\Lambda^c L\psi^* M_{V/X}]$$

in $G(Z_W)_{/\mathcal{L}_Z}$.

Proof of Proposition 2.40 ⇒ 2.16. 2. Let W be as in Proposition 2.16.2. Then we have $[[V, W]]_X = [L\Lambda^{c-1}M'_{V/X,W}]$ by Proposition 2.40 and it is in $F_{p-c}G(Z_D)_{/\mathcal{L}_Z}$ by Proposition 2.35. Further by Proposition 2.35, its class in $Gr^F_{p-c}G(Z_D)_{/\mathcal{L}_Z}$ is equal to the image of $(-1)^{c-1}c_{c-1}{}^D_{Z_D}(M'_{V/X,W}) \cap [D]$. Hence the assertion follows.

1. Let $W \to X$ be a scheme as in Proposition 2.16. For an integral closed subscheme W' of dimension p of W, let $W'' = W'$ and $\pi : W'' = W' \to W$ be the immersion if the image of W' in X is in V and let W'' be the blow-up of W' at $V_W = V \times_X W'$ and $\pi : W'' \to W' \to W$ be the composition if otherwise. As in the proof of Proposition 2.35, the topological filtration $F_pG(W)$ is generated by the class $\pi_*[O_{W''}]$ where W' runs integral closed subschemes of W of dimension $\leq p$. Hence it is sufficient to shows that $[[V, W']]_X$ is in the topological filtration $F_{p-c}G(Z_{W_V})$ assuming $\dim W' = p$. If the image of W' in X is in V, $W' = W''$ is a scheme over V and, if otherwise, the inverse image $T = V \times_X W''$ is a divisor of W''. Hence 2 shows that $[[V, W'']]_X$ is in the topological filtration $F_{p-c}G(Z_{W''_V})$. By the projection formula, Proposition 2.19, we have $[[V, W']]_X = \pi_*[[V, W'']]_X$. Thus it follows from Lemma 2.2.1.

We deduce Proposition 2.40, from the following Lemma.

Lemma 2.41 *Let the notation be as in Proposition 2.40. Then*

1. There exists a spectral sequence

$$E^2_{p,q} = L^p\Lambda^q M'_{V/X,W} \Rightarrow \mathcal{T}or_n^{O_X}(O_V, O_W)$$

of O_T-modules. If $W = T$ is a scheme over V, we have $E^2_{p,q} = L^p\Lambda^q\psi^* M_{V/X}$ and the spectral sequence degenerates at E^2-terms.

2. Let $i : Z \to X$ be the closed subscheme defined by the ideal $\mathrm{Ann}\Omega^n_{X/S}$. Let $N'_{V/X,W} = \mathcal{H}_0 M'_{V/X,W}$ be the excess conormal sheaf and let $i' : Z' \to T$ be the closed subscheme of T defined by the ideal $\mathrm{Ann}\Lambda^{c-c'} N'_{V/X,W}$. Then Z' is a subscheme of $Z_T = Z \times_X T$. We put $\mathcal{L}_Z = L^1 i^* L_{X/S}$ and $\mathcal{L}'_{Z'} = L^1 i'^* M'_{V/X,W}$. Then there is a canonical isomorphism $\mathcal{L}'_{Z'} \to \mathcal{L}_Z|_{Z'}$. The O_T-module $L^p\Lambda^q M'_{V/X,W}$ is an $O_{Z'}$-module for $p > 0$.

3. Let $X \to P$ be a regular closed immersion of codimension 1 and assume that the ideals $J \subset I \subset O_P$ defining X and V are generated by one section and m sections of O_P respectively.

We identify the excess conormal complex $M'_{V/X,W}$ with $[\varphi^* N_{X/P}|_T \to N_{V/X,W}]$. Then, the canonical maps $\iota_{M'_{V/X,W}} : L^p\Lambda^q M'_{V/X,W} \to L^{p-1}\Lambda^{q-1} M'_{V/X,W} \otimes N_{X/P}$ in Lemma 2.34.2 for $\mathcal{K} = M'_{V/X,W}$ and $\iota_{O_V,O_W,X/P} : \mathcal{T}or^{O_X}_{p+q}(O_V, O_W) \to \mathcal{T}or^{O_X}_{p+q-2}(O_V, O_W) \otimes N_{X/P}$ define a map of spectral sequences

$$(E^2_{p,q} \Rightarrow \mathcal{T}or^{O_X}_{p+q}(O_V, O_W)) \to (E^2_{p-1,q-1} \otimes N_{X/P} \Rightarrow \mathcal{T}or^{O_X}_{p+q-2}(O_V, O_W) \otimes N_{X/P}).$$

Corollary 2.42 *1. Assume $W = T$ is a scheme over V. Then there exist a decending filtration $F_p\mathcal{T}or^{O_X}_n(O_V, O_W)$ of O_W-modules on $\mathcal{T}or^{O_X}_n(O_V, O_W)$ satisfying $F_n = \mathcal{T}or^{O_X}_n(O_V, O_W)$ and $F_{-1} = 0$ and isomorphisms $L^p\Lambda^q L\varphi^* M_{V/X} \to Gr^F_p \mathcal{T}or^{O_X}_n(O_V, O_W)$.*

2. Let K be a discrete valuation field with perfect residue field and X be a regular and flat scheme of dimension n over $S =$ Spec O_K with smooth generic fiber. Then we have a decreasing filtration $F_p\mathcal{T}or^{O_{X\times_S X}}_n(O_X, O_X)$ of the O_X-module $\mathcal{T}or^{O_{X\times_S X}}_n(O_X, O_X)$ satisfying $F_n = \mathcal{T}or^{O_{X\times_S X}}_n(O_X, O_X)$ and $F_{-1} = 0$ and an isomorphism $L^p\Lambda^q\Omega^1_{X/S} \to Gr^F_p\mathcal{T}or^{O_{X\times_S X}}_n(O_X, O_X)$ for $p+q = n$. If Z denotes the closed subscheme of X defined by the ideal Ann $\Omega^n_{X/S}$*, the O_X-modules $L^p\Lambda^q\Omega^1_{X/S}$ are O_Z-modules for $p > 0$.*

Proof of Corollary 2.42. 1. The spectral sequence in Lemma 2.41.1 in the case W is a scheme over V degenerates at E^2-terms and defines the required filtration.

2. By Lemma 2.15.2, the conormal complex $M_{X/X\times_S X}$ is quasi-isomorphic to the conormal sheaf $N_{X/X\times_S X} = \Omega^1_{X/S}$. Hence, it suffices to apply 1 to the diagonal embedding $X \to X \times_S X$.

Proof of Lemma 2.41 ⇒ Proposition 2.40. By the spectral sequence in Lemma 2.41.1, we have $(-1)^m[\mathcal{T}or^{O_X}_m(O_V, O_W)] = \sum_{p+q=m}(-1)^{p+q}[E^\infty_{p,q}]$ for a sufficiently large integer m. Hence we have an equality $[[V, W]]_X = \sum_{p+q=m,m+1}(-1)^{p+q}[E^\infty_{p,q}]$ in $G(Z_W)_{/\mathcal{L}_Z}$. On the other hand, we have $(-1)^{c-c'}[L\Lambda^{c-c'} M'_{V/X,W}] = \sum_p(-1)^{p+c-c'}[E^2_{p,c-c'}]$. By Lemma 2.34.2 the right hand side is equal to $\sum_{p+q=m,m+1}(-1)^{p+q}[E^2_{p,q}]$ in $G(Z_W)_{/\mathcal{L}'_{Z'}}$. By Lemma 2.41.2, it implies the same equality in $G(Z_W)_{/\mathcal{L}_Z}$. Hence as in the proof of Proposition 2.23, it is reduced to the equality $\sum_{p+q=m,m+1}(-1)^{p+q}[E^\infty_{p,q}] = \sum_{p+q=m,m+1}(-1)^{p+q}[E^2_{p,q}]$ in $G(Z_W)_{/\mathcal{L}'_{Z'}}$. By the same argument as there, it is sufficient to show that the isomorphism $E^2_{p,q} = L^p\Lambda^q M'_{V/X,W} \to E^2_{p-1,q-1} \otimes \mathcal{L}'_{Z'}$ in Lemma 2.34.2 induces an isomorphism Im $d^r_{p,q} \to$ Im $d^r_{p-1,q-1} \otimes \mathcal{L}'_{Z'}$ if $p + q$ is sufficiently large. Similarly as in the proof of Lemma 2.31, it is proved using Lemma 2.41.3. The detail is left to the reader.

Proof of Lemma 2.41. 1. First we define a spectral sequence assuming the following conditions: There exists a regular closed immersion $X \to P$ of codimension l. Further the ideals $J \subset I \subset O_P$ defining X and V are generated by l sections $b_1, \cdots, b_l$ and m sections $a_1, \ldots, a_m$ of O_P respectively.

We define a generalized Koszul resolution of the O_X-module O_V locally on X. Take a matrix $M \in M_{ml}(O_P)$ satisfying $b = aM$ where $a = (a_1, \cdots, a_m) \in O_P^m$ and $b = (b_1, \cdots, b_l) \in O_P^l$. We put $\mathcal{L} = O_X^l, \mathcal{E} = O_X^m$ and define $f : \mathcal{L} \to \mathcal{E}$ and $a : \mathcal{E} \to O_X$ to be the multiplication by the image of M in $M_{ml}(O_X)$ and by the image of a in O_X^m respectively. The composite map $a \circ f : \mathcal{L} \to \mathcal{E} \to O_X$ is the 0-map. The O_X-module $\mathcal{L}$ is naturally identified with the conormal sheaf $N_{X/P}$.

Lemma 2.43 *1. The generalized Koszul complex* $\mathcal{G} = \mathrm{Kos}(\mathcal{K} \xrightarrow{a} O_X)$ *of the map* $\mathcal{K} = [\mathcal{L} \xrightarrow{f} \mathcal{E}] \xrightarrow{a} O_X$ *is a free resolution of the* O_X*-module* O_V*.*

2. Assume the codimension l *of the immersion* $X \to P$ *is 1. Then, the canonical map* $s\mathcal{G} \to s\mathcal{G} \otimes \mathcal{L}[2]$ *gives the canonical map* $\iota_{O_V, X/P} : O_V \to O_V \otimes N_{X/P}[2]$*.*

Proof of Lemma 2.43. 1. We consider the spectral sequence $E^1_{p,q} = \mathcal{H}_q(\mathcal{G}_{p,\bullet}) \Rightarrow \mathcal{H}_{p+q}(s\mathcal{G})$ where $s\mathcal{G}$ denotes the simple complex associated to $\mathcal{G}$. By the description of the second partial complex above, we have $Gr^F_p \mathcal{G} = \mathrm{Kos}(\mathcal{E} \xrightarrow{a} O_X) \otimes TS^p\mathcal{L}[p]$. The Koszul complex $\mathrm{Kos}(O_P^m \xrightarrow{a} O_P)$ is a free resolution of the O_P-module O_V. Since $\mathcal{T}or^{O_P}_{q-p}(O_V, O_X) = \Lambda^{q-p} N_{X/P}|_V$ by Lemma 2.6, we have a canonical isomorphism

$$E^1_{p,q} = \mathcal{H}_q(\mathcal{G}_{p,\bullet}) \to \mathcal{T}or^{O_P}_{q-p}(O_V, O_X) \otimes_{O_X} TS^p\mathcal{L} \to (\Lambda^{q-p} N_{X/P} \otimes_{O_X} TS^p N_{X/P})|_V.$$

It induces an isomorphism of complexes $(E^1_{\bullet,q}, d^1_{\bullet,q}) \to Lj^* L\Lambda^q(N_{X/P} \xrightarrow{\mathrm{id}} N_{X/P})$. Hence we have $E^2_{0,0} = O_V$ and $E^2_{p,q} = 0$ unless $p = q = 0$.

2. Recall that the map $s\mathcal{G} \to s\mathcal{G} \otimes \mathcal{L}[2]$ is identified with the canonical map $\mathrm{Cone}[s\mathcal{G} \otimes \mathcal{L}[1] \to \mathrm{Kos}(\mathcal{E} \to O_X)] \to s\mathcal{G} \otimes \mathcal{L}[2]$. Since the Koszul complex $\mathrm{Kos}(\mathcal{E} \to O_X)$ is quasi-isomorphic to $O_V \otimes^L_{O_P} O_X$, the assertion follows.

We go back to the proof of Lemma 2.41. We consider the spectral sequence

$$E^1_{p,q} = \mathcal{H}_q(\mathcal{G}_{p,\bullet} \otimes_{O_X} O_W) \Rightarrow \mathcal{H}_{p+q}(s\mathcal{G} \otimes_{O_X} O_W) = \mathcal{T}or^{O_X}_{p+q}(O_V, O_W) \tag{5}$$

of O_W-modules. The second partial complex $\mathcal{G}_{p,\bullet} \otimes_{O_X} O_W$ of the double complx $\mathcal{G} \otimes_{O_X} O_W$ is the same as $(\mathrm{Kos}(O_P^m \to O_P) \otimes_{O_P} O_W) \otimes_{O_W} \varphi^* TS^p N_{X/P}[p]$. Since $\mathrm{Kos}(O_P^m \to O_P)$ is a free resolution of O_V, we have $E^1_{p,q} = \mathcal{H}_q(\mathcal{G}_{p,\bullet} \otimes_{O_X} O_W) = \mathcal{T}or^{O_P}_{q-p}(O_V, O_W) \otimes_{O_W} \varphi^* TS^p N_{X/P}$. Thus we obtain a spectral sequence

$$E^1_{p,q} = \mathcal{T}or^{O_P}_{q-p}(O_V, O_W) \otimes_{O_W} \varphi^* TS^p N_{X/P} \Rightarrow \mathcal{T}or^{O_X}_{p+q}(O_V, O_W).$$

Assume the immersion $T = V \times_X W \to W$ is a regular immersion. Let $\varphi_T : T \to V$ be the natural map. We identify the excess conormal complex $M'_{V/X,W}$ with $[\varphi_T^*(N_{X/P}|_V) \to N'_{V/P,W}]$ where $N'_{V/P,W}$ is the excess conormal sheaf $\mathrm{Ker}(\varphi_T^* N_{V/P} \to N_{T/W})$. By Lemma 2.6, we have a canonical isomorphism $\mathcal{T}or^{O_P}_{q-p}(O_V, O_W) \to \Lambda^{q-p} N'_{V/P,W}$. It induces an isomorphism $E^1_{p,q} \to \Lambda^{q-p} N'_{V/P,W} \otimes_{O_T} \varphi_T^*(TS^p N_{X/P}|_V)$. It is easy to check that the isomorphisms and the identification above define an isomorphism $(E^1_{\bullet,q}, d^1_{\bullet,q}) \to L\Lambda^q M'_{V/X,W}$ of complexes. Hence we obtain a canonical isomorphism $E^2_{p,q} \to L^p \Lambda^q M'_{V/X,W}$.

Assume $W = T$ is a scheme over V. Then the conormal complex $M'_{V/X,W}$ is equal to the pull-back $L\psi^* M_{V/X}$. In this case, the second boundary maps of the double complex $\mathcal{G} \otimes_{O_X} O_W$ are the 0-maps. Hence the spectral sequence degenerates at E^2-terms.

We prove the general case. It is sufficient to show that the spectral sequence defined above are glued to define a global spectral sequence $E^2_{p,q} = L^p \Lambda^q M'_{V/X,W} \Rightarrow \mathcal{T}or^{O_X}_n(O_V, O_W)$ of O_W-modules. Let U and U' be open subschemes X and $U \to P$ and $U' \to Q$ be immersions into smooth schemes over S. Let l and l' be the codimensions of U in P and U' in Q respectively and

$m=c+l-1$ and $m'=c+l'-1$ be the codimensions of V_U in P and $V_{U'} \in Q$ respectively. Assume that the ideals $J \subset I \subset O_P$ defining U and V_U and the ideals $J' \subset I' \subset O_Q$ defining U' and $V_{U'}$ are generated by $b_1, \cdots, b_l$ and $a_1, \ldots, a_m \in O_P$ and $b'_1, \cdots, b'_{l'}$ and $a'_1, \ldots, a'_{m'} \in O_Q$ respectively. Take a matrix $M \in M_{ml}(O_P)$ satisfying $b = aM$ where $a = (a_1, \cdots, a_m) \in O_P^m, b = (b_1, \cdots, b_l) \in O_P^l$ and a matrix $M' \in M_{m'l'}(O_Q)$ satisfying $b' = a'M'$ where $a' = (a'_1, \cdots, a'_{m'}) \in O_Q^{m'}, b' = (b'_1, \cdots, b'_{l'}) \in O_Q^{l'}$. We define the generalized Koszul resolution $\mathcal{K} = \mathrm{Kos}([\mathcal{L} \xrightarrow{M} \mathcal{E}] \xrightarrow{a} O_U)$ of O_{V_U} and $\mathcal{K}' = \mathrm{Kos}([\mathcal{L}' \xrightarrow{M'} \mathcal{E}'] \xrightarrow{a'} O_{U'})$ of $O_{V_{U'}}$ as in Lemma 2.43.1. We consider spectral sequences $E^2_{p,q} = \mathcal{H}_p(\mathcal{H}_q(\mathcal{K}_{\bullet,\bullet} \otimes_{O_U} O_{W_U})) \Rightarrow \mathcal{T}or^{O_U}_{p+q}(O_{V_U}, O_{W_U})$ of O_{W_U}-modules and $E'^2_{p,q} = \mathcal{H}_p(\mathcal{H}_q(\mathcal{K}'_{\bullet,\bullet} \otimes_{O_{U'}} O_{W_{U'}})) \Rightarrow \mathcal{T}or^{O_{U'}}_{p+q}(O_{V_{U'}}, O_{W_{U'}})$ of $O_{W_{U'}}$-modules.

It is sufficient to define a canonical isomorphism of spectral sequences $E^2_{p,q}|_{T_{U\cap U'}} \to E'^2_{p,q}|_{T_{U\cap U'}}$ compatible with the identity of $\mathcal{T}or^{O_X}_{p+q}(O_V, O_W)|_{T_{U\cap U'}}$. By a standard product argument, we may assume $U = U' = X$ and there is a map $Q \to P$ compatible with the immersions $X \to P$ and $X \to Q$. We define a map of simple complexes $g : s\mathcal{G} \to s\mathcal{G}'$. Let $M_0 \in M_{m'm}(O_P)$ and $M_1 \in M_{l'l}(O_P)$ be matrices satisfying $a = a'M_0$ and $b = b'M_1$ respectively. Let $g_0 : \mathcal{E} \to \mathcal{E}'$ and $g_1 : \mathcal{L} \to \mathcal{L}'$ be the multiplication by the images of M_0 and of M_1 respectively. The diagrams

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & N_{V/P} \\ {\scriptstyle g_0}\downarrow & & \downarrow \\ \mathcal{E}' & \longrightarrow & N_{V/Q} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{L} & \longrightarrow & N_{X/P}|_V \\ {\scriptstyle g_1}\downarrow & & \downarrow \\ \mathcal{L}' & \longrightarrow & N_{X/Q}|_V \end{array}$$

are commutative. Since $f' : \mathcal{L}' \to \mathcal{E}'$ induces an isomorphism $N_{X/Q}|_V \to \mathcal{H}_1\mathrm{Kos}(\mathcal{E}' \to O)$, the image of $g_0 \circ f - f' \circ g_1 : \mathcal{L} \to \mathcal{E}'$ is contained in the image of $a'_2 : \Lambda^2\mathcal{E}' \to \mathcal{E}'$. Hence, we can define a map $g_2 : \mathcal{L} \to \Lambda^2\mathcal{E}'$ making the diagram

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{f} & \mathcal{E} & \longrightarrow & O_X \\ {\scriptstyle (g_1,g_2)}\downarrow & & {\scriptstyle g_0}\downarrow & & \| \\ \mathcal{L}' \oplus \Lambda^2\mathcal{E}' & \xrightarrow[(f',a'_2)]{} & \mathcal{E}' & \longrightarrow & O_X \end{array}$$

commutative. Applying Lemma 2.39, we find a quasi-isomorphism $g : s\mathcal{G} \to s\mathcal{G}'$ of filtered complexes. It induces an isomorphism of spectral sequences $E^1_{p,q} \to E'^1_{p,q}$ compatible with the identity $\mathcal{T}or^{O_X}_{p+q}(O_V, O_W)$. Since the map $(E^1_{\bullet,q}) \to (E'^1_{\bullet,q})$ of complexes is the same as the canonical quasi-isomorphism $L\Lambda^q(N_{X/P}|_V \to N'_{V/P,W}) \to L\Lambda^q(N_{X/Q}|_V \to N'_{V/Q,W})$, it induces the canonical isomorphism on the E^2-terms. Thus the assertion is proved.

2. We show $Z' \subset Z_T$. The assertion is local on X and on W. Shrinking X, we take a regular immersion $X \to P$ of codimension 1 into a smooth scheme P of relative dimension n over S. Shrinking X and W further, we take isomorphisms $O_X \to N_{X/P}, O_V^c \to N_{V/P}$, $O_X^n \to \Omega^1_{P/S}|_X$ and $O_T^{c-c'} \to N'_{V/P,W}$ and identify them. Let $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_c)$ and $(b'_1, \ldots, b'_{c-c'})$ be the image of 1 by the maps $N_{X/P} \to \Omega^1_{P/S}|_X$, $N_{X/P}|_V \to N_{V/P}$ and $\varphi_T^* N_{X/P}|_V \to N'_{V/P,W}$ respectively under this identification. Then the ideal $I_Z = \mathrm{Ann}\, \Omega^n_{X/S}$ is generated by $(a_1, \ldots, a_n)$ and the ideal $I_T = \mathrm{Ann}\, \Lambda^{c-c'} N'_{V/X,W}$ is generated by $(b'_1, \ldots, b'_{c-c'})$ respectively. Since the map

$N_{X/P}|_V \to \Omega^1_{P/S}|_V$ is factored by $N_{X/P}|_V \to N_{V/P}$ and the map $\varphi_T^* N_{X/P}|_V \to \varphi_T^* N_{V/P}$ is factored by $\varphi_T^* N_{X/P}|_V \to N_{V/P,W}$, we have inclusions $(a_1, \ldots, a_n) \subset (b_1, \ldots, b_c) \subset (b'_1, \ldots, b'_{c-c'})$ of ideals of O_T. Hence Z' is a subscheme of Z_T.

Since $Z' \subset Z_T$, the composition $M'_{X/S,W} \to L\varphi_T^* M_{V/X} \to L\varphi_T^* Lj^* L_{X/S}$ induces a map $\mathcal{L}'_{Z'} \to \mathcal{L}_Z|_{Z'}$. We show it is an isomorphism. The question is local on X. Shrinking X, let $X \to P$ be a regular immersion as above. Then both sheaves are identified with $(\varphi_T^*(N_{X/P}|_V))|_{Z'}$. Thus the assertion follows. By Lemma 2.34.1, the O_T-module $L^p\Lambda^q L\varphi_T^* M_{V/X,W}$ is an $O_{Z'}$-module for $p > 0$.

3. We show that the canonical map $\mathcal{G} \to \mathcal{G} \otimes \mathcal{L}[2]$ studied in Lemma 2.43.2 induces the required map. Clearly, the maps of E^2-terms are the canonical map $\iota_{M'_{V/X,W}} : L^p\Lambda^q M'_{V/X,W} \to L^{p-1}\Lambda^{q-1} M'_{V/X,W} \otimes N_{X/P}$. By the definition of the canonical map $\iota_{O_V,X/P}$ and by Lemma 2.43.2, the map on the limits are the canonical map $\iota_{O_V,O_W,X/P} : \mathcal{T}or^{O_X}_{p+q}(O_V, O_W) \to \mathcal{T}or^{O_X}_{p+q-2}(O_V, O_W) \otimes N_{X/P}$.

3. Logarithmic method.

We define and study the "logarithmic product", the "logarithmic diagonal map" and the sheaf of logarithmic differentials in §§3.1 and 3.2. We apply the K-theoretic localized intersection theory developed in §2 to the log diagonal map and define the "logarithmic localized intersection product" for regular schemes over a discrete valuation ring in §3.3. We also prove that the product is factored through the generic fiber, Proposition 3.20. We state a generalization, Theorem 3.25, of Theorem 1.15 to an algebraic correspondence in §3.4. Proof of Theorem 3.25 will be given in §4. In §3.5, we prove some of properties of log product which will be used in the proofs of Theorem 3.25 and of logarithmic Lefschetz trace formula. We state and prove logarithmic Lefschetz trace formula, Theorem 3.39, in §3.6. For generalities on log schemes such as the definitions of log smooth morphisms, exact immersions etc., we refer to [18], [20] and [15].

3.1. Log product.

Before defining logarithmic product, we briefly recall the definition of fs-log scheme. Basic references are [18], [20] and [23] Section 1. In this paper, we consider log structures defined with respect to the Zariski topology. In this paper, a monoid means a commutative monoid. For a monoid M, $M^{\rm gp}$ denotes the associated commutative group and $M^\times$ denotes the subgroup of invertible elements. A monoid M is called integral if the canonical map $M \to M^{\rm gp}$ is injective. We will identify an integral monoid M with its image in $M^{\rm gp}$. A monoid M is called saturated if it is integral and if it is equal to the saturation $M^{\rm sat} = \{x \in M^{\rm gp} | x^n \in M \text{ for some } n \geq 1\}$. A monoid is called an fs-monoid if it is finitely generated and saturated. An fs-log structure on a scheme X is a morphism $\alpha : M_X \to O_X$ of sheaves of monoids with respect to the multiplication on O_X satisfying the following conditions (1) and (2).

(1) The induced map $\alpha^{-1}(O_X^\times) \to O_X^\times$ is an isomorphism.

(2) For each point x, there exist an open neighboorhood U, an fs-monoid P and a morphism of monoids $\beta : P \to \Gamma(U, M_X)$ such that the diagram

$$\begin{array}{ccc} \beta^{-1}(M_X^\times|_U) & \xrightarrow{\subset} & P_U \\ \downarrow & & \downarrow{\scriptstyle \beta} \\ M_X^\times|_U & \xrightarrow{\subset} & M_X|_U \end{array}$$

is co-cartesian in the category of sheaves of monoids. Here P_U denotes the constant sheaf.

A morphism $\beta : P \to \Gamma(U, M_X)$ of monoids satisfying the condition above is called a chart of M_X on U. A scheme with an fs-log structure is called an fs-log scheme. In this paper, we only consider fs-log schemes and fs-log structures and we simply call an fs-log scheme a log scheme and an fs-log structure a log structure respectively. A log structure defined by $M_X = O_X^\times$ is called the trivial log structure. The condition (1) implies $M_X^\times = \alpha^{-1}(O_X^\times) \simeq O_X^\times$. For a log scheme X, we put $\bar{M}_X = M_X / M_X^\times$. The sheaf $\bar{M}_X$ of monoids satisfies the following properties.

Lemma 3.1 *Let X be an fs-log scheme.*

1. The maps $M_X \to M_X^{\mathrm{gp}}$ and $\bar{M}_X \to \bar{M}_X^{\mathrm{gp}}$ are injective and the diagram

$$\begin{array}{ccc} M_X & \longrightarrow & \bar{M}_X \\ \downarrow & & \downarrow \\ M_X^{\mathrm{gp}} & \longrightarrow & \bar{M}_X^{\mathrm{gp}} \end{array}$$

is cartesian.

2. The largest subgroup $\bar{M}_X^{\times}$ is trivial.

Proof. It is sufficient to prove stalkwise. Let x be a point. By [20] Lemma 1.6, there exists a non-canonical isomorphism of monoids $M_{X,x} \to \bar{M}_{X,x} \times O_{X,x}^{\times}$ and $\bar{M}_{X,x}$ is an fs-monoid satisfying $\bar{M}_{X,x}^{\times} = 1$. The assertions follow from this immediately.

Let $X \to Y$ be a map of fs-log schemes and let $\tilde{M} \to M$ be a surjection of fs-monoids satisfying $\mathrm{Ker}(\tilde{M} \to M) \subset \tilde{M}^{\times}$. When we have a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & M \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\bar{\theta}} \\ \Gamma(X, M_X) & \longrightarrow & \Gamma(Y, \bar{M}_Y), \end{array}$$

we say $\bar{\theta}$ is induced by θ or θ is a lifting of $\bar{\theta}$. By Lemma 3.1.2, the map $\bar{\theta}$ is uniquely determined by θ.

In order to construct log products, we introduce the notion of frames of log schemes. The following criterion will be used in the definition of frames.

Lemma 3.2 *1. Let M and N be fs-monoids and assume $N^{\times} = 1$. For a surjective morphism $\theta : M \to N$ of monoids, the following conditions are equivalent.*

(1) For $a, b \in M$, we have $\theta(a) = \theta(b)$ if and only if there exist $u, v \in M$ satisfying $au = bv$ and $\theta(u) = \theta(v) = 1$.

(2) Put $S = \mathrm{Ker}\theta$ and $M' = S^{-1}M$. Then the induced map $M'/M'^{\times} \to N$ is an isomorphism.

2. If $M = \mathbf{N}^r$, the conditions (1) and (2) are further equivalent to the following condition.

(3) There exists a subset $I \subset \{1, \ldots, m\}$ and an isomorphism $\bar{\theta} : \mathbf{N}^I \to N$ of monoids such that θ is the composition $\bar{\theta} \circ pr_I$ with the projection.

Proof. 1. (1)⇒(2). It is sufficient to show the injectivity. Assume $\theta(a/s) = \theta(b/t)$ for $a, b \in M$ and $s, t \in S$. Then there are $u, v \in M$ such that $atu = bsv$ and $\theta(u) = \theta(v) = 1$ by (1). Thus we have $a/s = b/t \cdot v/u$ and $v/u \in M'^{\times}$.

(2)⇒(1). Assume $\theta(a) = \theta(b)$ for $a, b \in M$. By (2), there are $u \in M$ and $v \in S$ such that $u/v \in M'^{\times}$ and $au/v = b$. Since N is integral and $N^{\times} = 1$, we have $\theta(u) = \theta(v) = 1$

2. (3)⇒(2). Clear.

(2)⇒(3). Let e_i be the standard basis of $M = \mathbf{N}^m$. Then the kernel $S = \mathrm{Ker}\ \theta$ is generated by e_i such that $\theta(e_i) = 0$. Thus the assertion follows.

Definition 3.3 *1. Let X be a log scheme and M be an fs-monoid. A morphism of monoids $M \to \Gamma(X, \bar{M}_X)$ is called a frame if the following condition is satisfied.*

For each point $x \in X$, the map $M \to \bar{M}_{X,x}$ satisfies the equivalent conditions in Lemma 3.2.

2. Let $X \to Y$ be a morphism of log schemes and $M \to \Gamma(X, \bar{M}_X)$ and $N \to \Gamma(Y, \bar{M}_Y)$ be frames. A map of monoids $N \to M$ is said to be a map of frames if the diagram

$$\begin{array}{ccc} N & \longrightarrow & M \\ \downarrow & & \downarrow \\ \Gamma(Y, \bar{M}_Y) & \longrightarrow & \Gamma(X, \bar{M}_X) \end{array}$$

is commutative. Similarly, we define a map of charts by replacing $\bar{M}_X$ and $\bar{M}_Y$ by M_X and M_Y respectively.

For an fs-monoid M, we regard $S = \operatorname{Spec} \mathbf{Z}[M]$ as a log scheme with the log structure associated to $M \to \mathbf{Z}[M]$. The canonical map $M \to \Gamma(S, \bar{M}_S)$ is a frame. Typical examples of frames are produced by the following Lemma. For a regular scheme X with a divisor D with simple normal crossings, its standard log structure is defined by $M_X = O_X \cap j_* O_U^\times$ where $j : U \to X$ is the open immersion of the complement of D.

Lemma 3.4 *1. Let X be a regular scheme with a divisor D with simple normal crossings. If $D_1, \ldots, D_m$ are the irreducible components of D, the monoid $M = \Gamma(X, \bar{M}_X)$ is isomorphic to $\mathbf{N}^m$ and the identity $M \to \Gamma(X, \bar{M}_X)$ is a frame of X.*

2. Let X be a log regular log scheme and suppose $\mathbf{N}^m \to \Gamma(X, \bar{M}_X)$ is a frame. Then X is regular and M_X is the standard log structure associated to a divisor with simple normal crossings.

The frame in Lemma 3.4.1 is called the standard frame of X. Proof will be given using the following Proposition on a relation between a frame and a chart.

Proposition 3.5 *Let M be an fs-monoid and X be a log scheme. For a morphism $\theta : M \to \Gamma(X, \bar{M}_X)$ of monoids, the following conditions are equivalent.*

(1) $\theta : M \to \Gamma(X, \bar{M}_X)$ is a frame of X.

(2) For each point $x \in X$, there exists an open neighborhood U of x and a chart $M \to \Gamma(U, M_X)$ of U lifting $\theta : M \to \Gamma(X, \bar{M}_X)$.

We call a chart $M \to \Gamma(U, M_X)$ as in the condition (2) a local chart lifting the frame $M \to \Gamma(X, \bar{M}_X)$.

Proof of Proposition 3.5. It is sufficient to show the following Lemma.

Lemma 3.6 *1. Let $x \in X$ be a point and $\bar{\theta} : M \to \bar{M}_{X,x}$ be a morphism of fs-monoids. Then there exists an open neighborhood U of x and a lifting $M \to \Gamma(U, M_X)$ of $\bar{\theta}$.*

2. Let $M \to \Gamma(X, M_X)$ be a morphism of monoid. Then the composition $M \to \Gamma(X, M_X) \to \Gamma(X, \bar{M}_X)$ is a frame if and only if $M \to \Gamma(X, M_X)$ is a chart.

Proof. 1. Since $\bar{M}^{\times}_{X,x} = 1$, we may assume $M^{\times}$ is trivial. Since M is an fs-monoid, $M^{\rm gp}$ is a free abelian group. Hence there exists an open neighborhood U of x and a lifting $M^{\rm gp} \to \Gamma(U, M^{\rm gp}_X)$ of $\bar{\theta}^{\rm gp} : M^{\rm gp} \to \bar{M}^{\rm gp}_{X,x}$. The image of $M \subset M^{\rm gp}$ in $M^{\rm gp}_{X,x}$ is contained in $M_{X,x}$. Shrinking U if necessary, the image of M is contained in $\Gamma(U, M_X)$.

2. Let M'_X be the log structure associated to the composition map $\alpha : M \to O_X$ and let $M'_X \to M_X$ be the induced map. It is sufficient to show that $M \to \Gamma(X, \bar{M}_X)$ is a frame if and only if the map $M'_X \to M_X$ is an isomorphism. The latter condition is equivalent to that the map $\bar{M}'_{X,x} \to \bar{M}_{X,x}$ is an isomorphism for each point x of X. The sheaf M'_X is the push-out of the constant sheaf associated to M and $O^{\times}_X$ over $\alpha^{-1}(O^{\times}_X)$. Hence, for each $x \in X$, $\bar{M}'_{X,x}$ is identified with $M''_x/M''^{\times}_x$ where $M''_x = (\alpha_x^{-1}(O^{\times}_{X,x}))^{-1}M$. Thus the assertion is proved.

Proof of Lemma 3.4. (1) $\Rightarrow$ (2). Since $\bar{M}_X$ is the direct sum of the constant sheaves $\mathbf{N}_{D_i}$ on D_i, the first assertion is clear. For each $x \in X$, there exists an open neighborhood U of x and $t_i \in \Gamma(U, O_X)$ for $i = 1, \ldots, m$ which is a uniformizer of $D_i \cap U$ if $x \in D_i$ and is a unit if $x \notin D_i$. The map $\mathbf{N}^m \to \Gamma(U, M_X)$ sending the standard basis e_i to t_i is a local chart lifting the identity $M \to \Gamma(X, \bar{M}_X)$. Hence by Proposition 3.5 (2) $\Rightarrow$ (1), the identity $M \to \Gamma(X, \bar{M}_X)$ is a frame.

(2) $\Rightarrow$ (1). By Proposition 3.5 (1) $\Rightarrow$ (2), we obtain a chart $\mathbf{N}^m \to \Gamma(U, M_X)$ on an open neighborhood U of each point x. Since X is log regular, it immediately follows from the definition, [20] Definition (2.1), that the scheme X is regular and M_X is the standard log structure associated to a divisor with simple normal crossings.

Corollary 3.7 *1. Let $M \to \Gamma(X, \bar{M}_X)$ be a map of monoid. Then $M \to \Gamma(X, \bar{M}_X)$ is a frame if and only if the induced map $M/M^{\times} \to \Gamma(X, \bar{M}_X)$ is a frame.*

2. Let $X \to Y$ be a morphism of log schemes, $M \to \Gamma(X, \bar{M}_X), N \to \Gamma(Y, \bar{M}_Y)$ be frames and $N \to M$ be a morphism of frames. For $x \in X$, there exist an open neighborhood U of x and V of $y = f(x)$ and charts $M + N^{\rm gp} \to \Gamma(U, M_X), N \to \Gamma(V, M_Y)$ lifting the frames $M \to \Gamma(X, \bar{M}_X), N \to \Gamma(Y, \bar{M}_Y)$ such that the map $N \to M + N^{\rm gp}$ is a morphism of charts.

3. Let $X \to S$ and $Y \to S$ be morphisms of log schemes, $M \to \Gamma(X, \bar{M}_X), N \to \Gamma(Y, \bar{M}_Y)$ and $L \to \Gamma(S, \bar{M}_S)$ be frames and $L \to M$ and $L \to N$ be maps of frames. Then the induced map $M +^{\rm sat}_L N \to \Gamma(X \times^{\log}_S Y, \bar{M}_{X \times^{\log}_S Y})$ is also a frame.

4. Let $X \to Y$ be a morphism of log schemes and $M \to \Gamma(Y, \bar{M}_Y)$ be a frame. The map $X \to Y$ is strict if and only if the composition $M \to \Gamma(X, \bar{M}_X)$ is also a frame.

In 3, $X \times^{\log}_S Y$ denotes the fibered product as an fs-log scheme and $M +^{\rm sat}_L N$ denotes the saturation of the image of $M + N$ in $(M^{\rm gp} \oplus N^{\rm gp})/L^{\rm gp}$ and is an fs-monoid.

Proof. 1. It follows immediately from the definition.

2. The question is local on X and Y by Proposition 3.5. Shrinking them if necessary, we may assume there are charts $M \to \Gamma(X, M_X)$ and $N \to \Gamma(Y, M_Y)$ lifting the frames by Proposition 3.5. Since the diagram

$$\begin{array}{ccc} N & \longrightarrow & M \\ \downarrow & & \downarrow \\ \Gamma(Y, M_Y) & \longrightarrow & \Gamma(X, \bar{M}_X) \end{array}$$

is commutative, the difference defines a map $N^{\rm gp} \to \Gamma(X, M_X^\times)$ making the diagram

$$\begin{array}{ccc} N & \longrightarrow & M + N^{\rm gp} \\ \downarrow & & \downarrow \\ \Gamma(Y, M_Y) & \longrightarrow & \Gamma(X, M_X) \end{array}$$

commutative. The map $M + N^{\rm gp} \to \Gamma(X, M_X)$ is a chart and the assertion follows.

3. The question is local on X, Y and S. Shrinking them if necessary, we may assume there are charts $M + L^{\rm gp} \to \Gamma(X, M_X), N + L^{\rm gp} \to \Gamma(Y, M_Y)$ and $L \to \Gamma(S, M_S)$ lifting the frames such that the maps $L \to M + L^{\rm gp}$ and $L \to N + L^{\rm gp}$ are maps of charts by 2. Since the natural map $(M + L^{\rm gp}) +_L^{\rm sat} (N + L^{\rm gp}) \to M +_L^{\rm sat} N$ induces an isomorphism $((M + L^{\rm gp}) +_L^{\rm sat} (N + L^{\rm gp}))/((M + L^{\rm gp}) +_L^{\rm sat} (N + L^{\rm gp}))^\times \to (M +_L^{\rm sat} N)/(M +_L^{\rm sat} N)^\times$, by replacing M by $M + L^{\rm gp}$ and N by $N + L^{\rm gp}$, we may assume that $L \to M$ and $L \to N$ are morphisms of charts. Then we have $X \times_S^{\log} Y = (X \times_S Y) \otimes_{\mathbf{Z}[M+N]} \mathbf{Z}[M +_L^{\rm sat} N]$ and the natural map $M +_L^{\rm sat} N \to \Gamma(X \times_S^{\log} Y, M_{X \times_S^{\log} Y})$ is a chart. Thus it follows from Proposition 3.5.

4. The question is local on X and on Y. Let $M \to \Gamma(Y, M_Y)$ be a chart lifting the frame $M \to \Gamma(Y, \bar{M}_Y)$. If $X \to Y$ is strict, the composition $M \to \Gamma(Y, M_Y) \to \Gamma(X, M_X)$ is a chart and it induces a frame $M \to \Gamma(X, \bar{M}_X)$. Conversely, assume the composition $M \to \Gamma(X, \bar{M}_X)$ is a frame. Then by 2, we find a chart $M + M^{\rm gp} \to \Gamma(X, M_X)$ lifting the frame $M \to \Gamma(X, \bar{M}_X)$ such that the diagonal map $M \to M + M^{\rm gp}$ is a morphism of charts. Therefore the log structure M_X is the pull-back of M_Y.

Proposition 3.8 *Let X be an fs-log scheme, $M \to \Gamma(X, \bar{M}_X)$ be a frame and $M \to N$ be a morphism of monoids such that the induced map $M^{\rm gp} \to N^{\rm gp}$ is a surjection. Then the functor associating to an fs-log scheme T a set*

$$\{f \in X(T) |\ \textit{the composite } M \to \Gamma(X, \bar{M}_X) \to \Gamma(T, \bar{M}_T) \textit{ is factored through } M \to N\}$$

is representable by a log scheme $\tilde{X}$ log etale over X. The tautological map $N \to \Gamma(\tilde{X}, \bar{M}_{\tilde{X}})$ is a frame.

A similar construction is studied in [16]. When we want to specify M and N, by abuse of notation, we will write $\tilde{X}$ by $X +_M N$.

Proof. If a map $M \to \Gamma(X, \bar{M}_X) \to \Gamma(T, \bar{M}_T)$ is factored through $M \to N$, the induced map $N \to \Gamma(T, \bar{M}_T)$ is unique. In fact, since $\Gamma(T, \bar{M}_T)$ is integral, the map $N \to \Gamma(T, \bar{M}_T)$ is determined by $N^{\rm gp} \to \Gamma(T, \bar{M}_T^{\rm gp})$. Hence the assertion is local on X. We may assume that there is a chart $M \to \Gamma(X, M_X)$ lifting the frame $M \to \Gamma(X, \bar{M}_X)$. Let M' denote the inverse image of N by the map $M^{\rm gp} \to N^{\rm gp}$. We show that the log fiber product $\tilde{X} = X \times_{\mathbf{Z}[M]}^{\log} \mathbf{Z}[M']$ represents the functor. The log scheme $\tilde{X}$ represents the functor

$$T \mapsto \{f \in X(T) |\ \text{the composite } M \to \Gamma(X, M_X) \to \Gamma(T, M_T) \text{ is factored through } M \to M'\}$$

since the map $M' \to \Gamma(T, M_T)$ is unique if it exists as above. Hence, it is sufficient to show that the map $M \to \Gamma(X, \bar{M}_X) \to \Gamma(T, \bar{M}_T)$ is factored by $M \to N$ if and only if the map

$M \to \Gamma(X, M_X) \to \Gamma(T, M_T)$ is factored by $M \to M'$. If part follows from the cartesian diagram in Lemma 3.1.1 and the only if part follows from $\bar{M}_T^\times = 1$ in Lemma 3.1.2.

Since $M' \to \Gamma(\tilde{X}, M_{\tilde{X}})$ is a chart, the induced map $N \to \Gamma(\tilde{X}, \bar{M}_{\tilde{X}})$ is a frame by Lemma 3.6.2 and Corollary 3.7.1.

Corollary 3.9 *Let $f : X \to S$ and $g : Y \to S$ be log schemes over a log scheme S, $M \to \Gamma(X, \bar{M}_X), M \to \Gamma(Y, \bar{M}_Y)$ and $N \to \Gamma(S, \bar{M}_S)$ be frames and $N \to M$ be a map of frames with respect to both of X and Y. Then on the category of log schemes, the functor associating to a log scheme T the set*

$$\left\{ (f,g) \in X(T) \times_{S(T)} Y(T) \,\middle|\, \text{the diagram} \begin{array}{ccc} M & \longrightarrow & \Gamma(X, \bar{M}_X) \\ \downarrow & & \downarrow f^* \\ \Gamma(Y, \bar{M}_Y) & \underset{g^*}{\longrightarrow} & \Gamma(T, \bar{M}_T) \end{array} \text{is commutative} \right\}$$

is representable by a log scheme $(X \times_S Y)^\sim$ log etale over the log fiber product $X \times_S^{\log} Y$. The projections $pr_1 : (X \times_S Y)^\sim \to X, pr_2 : (X \times_S Y)^\sim \to Y$ are strict.

Proof. By Corollary 3.7.3, the induced map $M +_N^{\text{sat}} M \to \Gamma(X \times_S^{\log} Y, \bar{M}_{X \times_S^{\log} Y})$ is a frame. We apply Proposition 3.8 to this frame and to the surjection $+ : M +_N^{\text{sat}} M \to M$ in order to define a log scheme $(X \times_S Y)^\sim = (X \times_S^{\log} Y) +_{M +_N^{\text{sat}} M} M$. Then the log scheme $(X \times_S Y)^\sim$ represents the functor and log etale over the log fiber product $X \times_S^{\log} Y$. Since the tautological map $M \to \Gamma((X \times_S X)^\sim, \bar{M}_{(X \times_S X)^\sim})$ is a frame, the projections are strict by Corollary 3.7.4.

We call $(X \times_S Y)^\sim$ the log fiber product of X and Y over S with respect to the map $N \to M$ of frames. Taking local charts lifting the frames, the log product is described explicitly as follows.

Corollary 3.10 *Let X and Y be log schemes over a log scheme S, $M \to \Gamma(X, \bar{M}_X), M \to \Gamma(Y, \bar{M}_Y)$ and $N \to \Gamma(S, \bar{M}_S)$ be frames and $N \to M$ be a map of frames both for X and Y. Let U and V be open subschemes of X and Y respectively, $M^\sim \to M$ be morphisms of fs-monoid inducing an isomorphism $M^\sim / M^{\sim\times} \to M/M^\times$. Let $M^\sim \to \Gamma(U, M_X), M^\sim \to \Gamma(V, M_Y)$ and $N \to \Gamma(S, M_S)$ be charts lifting the frames $M \to \Gamma(U, \bar{M}_X), M \to \Gamma(V, \bar{M}_Y)$ and $N \to \Gamma(S, \bar{M}_S)$. Let $N \to M^\sim$ be a map of charts both for U and V inducing the map of frames $N \to M$. Then there is a canonical isomorphism*

$$(X \times_S Y)^\sim \times_{X \times_S Y} (U \times_S V) = (U \times_S V)^\sim \to (U \times_S V) \otimes_{\mathbf{Z}[M^\sim + M^\sim]} \mathbf{Z}[M^\sim + (M^{\sim \text{gp}}/N^{\text{gp}})]$$

where the map $M^\sim + M^\sim \to M^\sim + (M^{\sim \text{gp}}/N^{\text{gp}})$ is given by $(a,b) \mapsto (a+b, a)$.

In Corollary, $+$ denotes the direct sum of monoids.

Proof. We may assume $X = U, Y = V$ and $M^\sim = M$. Let $(M +_N M)^\sim$ be the inverse image of M by the map $+ : (M +_N^{\text{sat}} M)^{\text{gp}} \to M^{\text{gp}}$. By the proofs of Proposition 3.8 and of Corollary 3.9, we have

$$(X \times_S Y)^\sim = (X \times_S^{\log} Y) \otimes_{\mathbf{Z}[M +_N^{\text{sat}} M]} \mathbf{Z}[(M +_N M)^\sim] = (X \times_S Y) \otimes_{\mathbf{Z}[M+M]} \mathbf{Z}[(M +_N M)^\sim].$$

Therefore, it is sufficient to show that the map $M + M \to M + (M^{\rm gp}/N^{\rm gp}) : (a, b) \mapsto (a + b, a)$ induces an isomorphism $(M +_N M)^\sim \to M + (M^{\rm gp}/N^{\rm gp})$. Since, it induces an isomorphism $M^{\rm gp} +_{N^{\rm gp}} M^{\rm gp} = (M +^{\rm sat}_N M)^{\rm gp} \to M^{\rm gp} \oplus (M^{\rm gp}/N^{\rm gp})$ of abelian groups, the claim follows.

We may describe the log blow-up associated to a cone decomposition in terms of the construction above in the following way. Let M be an fs-monoid and $N = Hom_{\rm monoid}(M, \mathbf{N})$ be the dual monoid. Let $\Delta \not\ni 0$ be a finite subset of N such that the saturation of the submonoid generated by Δ in $N^{\rm gp}$ is N. We consider Δ as the set of vertices. For a subset $\sigma \subset \Delta$, let N_σ be the saturation of the submonoid of N generated by σ and put $M_\sigma = Hom_{\rm monoid}(N_\sigma, \mathbf{N}) = \{x \in M^{\rm gp} | \langle x, v\rangle \geq 0 \text{ for } v \in \sigma\}$. For $\sigma \subset \Delta$ and $x \in M_\sigma$, we put $\sigma_x = \{v \in \sigma | \langle x, v\rangle = 0\}$. We say a set Σ of subsets of Δ is a decomposition of N if the following conditions are satisfied.

1. For $\sigma \in \Sigma$, $N_\sigma \cap \Delta = \sigma$.
2. For $v \in \Delta$, $\{v\} \in \Sigma$.
3. For $\sigma \in \Sigma$ and $x \in M_\sigma$, $\sigma_x \in \Sigma$.
4. For $\sigma, \tau \in \Sigma$, there exists $x \in M_\sigma$ such that $\sigma \cap \tau = \sigma_x$.
5. $N = \bigcup_{\sigma\in\Sigma} N_\sigma$.

If N_σ for $\sigma \in \Sigma$ are isomorphic to monoids of the form $\mathbf{N}^r$, we say Σ is regular.

Let $M \to \Gamma(X, \bar{M}_X)$ be a frame of a log scheme and Σ be a cone decomposition of $N = Hom_{\rm monoid}(M, \mathbf{N})$. Then we define the scheme X_Σ as follows. For each $\sigma \in \Sigma$, let X_σ be the log scheme $X +_M M_\sigma$. If $\sigma, \tau \in \Sigma$ and if $\sigma \subset \tau$, the natural map $X_\sigma \to X_\tau$ is an open immersion. Patching X_σ and X_τ at $X_{\sigma\cap\tau}$ for $\sigma, \tau \in \Sigma$, we obtain X_Σ.

Lemma 3.11 *Let $M \to \Gamma(X, \bar{M}_X)$ be a frame of a log scheme and Σ be a cone decomposition of the dual monoid $N = Hom_{\rm monoid}(M, \mathbf{N})$. Then the map $X_\Sigma \to X$ is proper and log etale.*

If X is log regular and if Σ is regular, the scheme X_Σ is regular and the log structure on X_Σ is defined by a divisor with normal crossings.

Proof. It is well-known that the map $X_\Sigma \to X$ is proper (cf. [20] Proposition (9.11)). Since $X_\sigma \to X$ is log-etale for $\sigma \in \Sigma$, $X_\Sigma \to X$ is log etale. Since $M_\sigma \to \Gamma(X_\sigma, \bar{M}_{X_\sigma})$ is a frame, the last assertion follows from the assumption that Σ is regular and Lemma 3.4.2.

3.2. Log diagonal and log differentials

Using the construction in §3.1, we define the log diagonal map. We also study its relation with the sheaf of logarithmic differentials.

Definition 3.12 *Let $f : X \to S$ be a morphism of log schemes.*

1. We define the sheaf of log differential forms $\Omega^1_{X/S}(\log/\log)$ by

$$\Omega^1_{X/S}(\log/\log) = (\Omega^1_{X/S} \oplus O_X \otimes_{\mathbf{Z}} (M^{\rm gp}_X/f^*M^{\rm gp}_S))/((d\alpha(m), -\alpha(m) \otimes m) : m \in M_X).$$

For $m \in M_X$, we put $d\log m = 1 \otimes m$.

2. Assume that $M = \Gamma(X, \bar{M}_X), N = \Gamma(S, \bar{M}_S)$ are fs-monoids and that the identity maps $M \to \Gamma(X, \bar{M}_X), N \to \Gamma(S, \bar{M}_S)$ define frames on X and S. We call the frames $M \to \Gamma(X, \bar{M}_X), N \to \Gamma(S, \bar{M}_S)$ the standard frames. We define the log self-product $(X \times_S X)^\sim$ to be the log product

$(X \times_S X)^\sim$ *constructed in Corollary 3.9 with respect to the canonical map* $N \to M$ *of the standard frames. The log diagonal map*

$$\Delta^\sim : X \to (X \times_S X)^\sim$$

is the map corresponding to the pair $(\mathrm{id}_X, \mathrm{id}_X)$.

The sheaf of log differential forms is the same as the conormal sheaf of the log diagonal.

Proposition 3.13 *Let* $X \to S$ *be a morphism of log schemes. Assume that* $M = \Gamma(X, \bar{M}_X), N = \Gamma(S, \bar{M}_S)$ *are fs-monoids and that the identity maps* $M \to \Gamma(X, \bar{M}_X), N \to \Gamma(S, \bar{M}_S)$ *are frames. Then, there is a canonical isomorphism*

$$\Omega^1_{X/S}(\log / \log) \to N_{X/(X \times_S X)^\sim}.$$

Proof. Let X_1 be the scheme affine over X corresponding to the O_X-algebra $O_X \oplus \Omega^1_{X/S}(\log / \log)$ where we define the multiplication by $\Omega^1_{X/S}(\log / \log)^2 = 0$. We regard X_1 as a log scheme with the inverse image log structure of that on X. The projection $p_1 : X_1 \to X$ is strict. The map $\Omega^1_{X/S}(\log / \log) \to 0$ defines a section $i_1 : X \to X_1$. The square of the ideal defining the section is 0. Let X_2 be the closed subscheme of $(X \times_S X)^\sim$ defined by the square $I^2_{\Delta^\sim_X}$ of the ideal defining the log diagonal immersion $\Delta^\sim_X : X \to (X \times_S X)^\sim$. We regard X_2 as a log scheme with the inverse image log structure of that on $(X \times_S X)^\sim$. The projection $p_2 : X_2 \to X$ is strict. The log diagonal defines a section $i_2 : X \to X_2$. The square of the ideal defining the section is 0. Since the conormal sheaves are $N_{X/X_1} = \Omega^1_{X/S}(\log / \log)$ and $N_{X/X_2} = N_{X/(X \times_S X)^\sim}$, it is reduced to define an isomorphism $X_2 \to X_1$ compatible with the immersions $X \to X_1, X \to X_2$.

To prove it, we use the following notation. For a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

of schemes, let $Hom^A_D(B, C)$ denote the set of morphisms $B \to C$ of schemes which make the two triangles commutative. When the diagram above is a commutative diagram of log schemes, we similarly define a set $Hom^{\log}{}^A_D(B, C)$ of morphisms of log schemes.

To define an isomorphism, it is sufficient to give a functorial bijection $Hom^{\log}{}^X_X(T, X_2) \to Hom^{\log}{}^X_X(T, X_1)$ for a log scheme T over X together with a section $i_T : X \to T$ such that the square of the ideal $\mathcal{I}_T$ defining the section i_T is 0 and that the projection $p_T : T \to X$ is strict. If we identify the underlying sets of X and T, the natural map $\bar{M}_T \to \bar{M}_X$ is an isomorphism.

First, we identify the set $Hom^{\log}{}^X_X(T, X_1)$. We regard the ideal $\mathcal{I}_T$ as an O_X-module. Since the projections are strict, we have a natural identification

$$Hom^{\log}{}^X_X(T, X_1) = Hom^X_X(T, X_1) = Hom_{O_X}(\Omega^1_{X/S}(\log / \log), \mathcal{I}_T).$$

On the other hand, we identify the set $Hom^{\log}{}^X_X(T, X_2)$ as follows. Since $\mathcal{I}_T^2 = 0$, we have $Hom^{\log}{}^X_X(T, X_2) = Hom^{\log}{}^X_X(T, (X \times_S X)^\sim)$. We show that the natural maps

$$Hom^{\log}{}^X_X(T, (X \times_S X)^\sim) \longrightarrow Hom^{\log}{}^X_X(T, X \times^{\log}_S X) \longrightarrow Hom^{\log}{}^X_S(T, X)$$

are bijections. The first map is a bijection since the natural map $\bar{M}_T \to \bar{M}_X$ is an isomorphism. The second map is simply the universality of the fiber product. By Lemma 3.14 below and the bijection $Hom_S^X(T,X) \to Hom_{O_X}(\Omega^1_{X/S}, \mathcal{I}_T)$, we have a functorial bijection

$$\begin{array}{ccc} Hom^{\log}{}_S^X(T,X) & \longrightarrow & Hom_{O_X}(\Omega^1_{X/S}(\log/\log), \mathcal{I}_T) \\ f & \mapsto & \begin{cases} adb \mapsto a(f^*b - b) & \text{for } a, b \in O_X, \\ d\log m \mapsto t & \text{for } m \in M_X \end{cases} \end{array}$$

where for $m \in M_X$, the section $t \in \mathcal{I}_T$ is characterized by $f^*m = m(1+t)$ as in Lemma 3.14.

Lemma 3.14 *The map $M_T \to M_X$ of sheaves of monoids is surjective. Let m, m' be sections on M_T on a open set U of X. If their images in M_X are equal, there exists a unique section t of $\mathcal{I}_T$ on U such that $m' = m(1+t)$.*

Proof. Straightforward and left to the reader.

3.3. Logarithmic localized intersection product.

We will define logarithmic localized intersection product as the localized intersection product with the log diagonal. We prove that the logarithmic localized intersection product is factered by the restriction to the generic fiber, Proposition 3.20.

In this subsection, we use the notation in §1. Namely, K is a discrete valuation field with perfect residue field, $S = \text{Spec } O_K$ and s is the closed point of S. X is a regular and flat scheme of finite type of dimension $n = \dim X$ over S such that the generic fiber of X is smooth and the reduced closed fiber has simple normal crossings. We consider X as a log scheme with the standard log structure M_X in Lemma 3.4 defined by the reduced closed fiber $X_{s,\text{red}}$. We put $M = \Gamma(X, \bar{M}_X)$. If $D_1, \ldots, D_m$ are the irreducible components of X_s, the monoid M is identified with $\mathbf{N}^m$. The identity $M \to \Gamma(X, \bar{M}_X)$ is the standard frame of X. We also consider $S = \text{Spec } O_K$ as a log scheme with the standard log structure defined by the closed point unless otherwise stated. We put $N = \Gamma(S, \bar{M}_S)$ and identify $N = \mathbf{N}$. If the multiplicity of D_i in X_s is l_i, the canonical map $N = \mathbf{N} \to M = \mathbf{N}^m$ sends 1 to $(l_1, \ldots, l_m)$. The log product $(X \times_S X)^\sim$ is defined to be that with respect to the canonical map $N \to M$ of frames. The sheaf $\Omega^1_{X/S}(\log/\log)$ defined in §1.3 is the same as that defined in Definition 3.12.1.

In the following, we regard the log product $(X \times_S X)^\sim$ as a scheme over X with respect to the second projection. In order to define the logarithmic localized intersection product, we show that $(X \times_S X)^\sim$ is locally a hypersurface over X. If U is an open subscheme of X, the log product $(U \times_S X)^\sim$ with respect to the frame $M = \Gamma(X, \bar{M}_X) \to \Gamma(U, \bar{M}_X)$ is naturally identified with the fiber product $U \times_X (X \times_S X)^\sim$. Let $i : U \to P$ be a regular immersion of codimension 1 into a smooth scheme P over S as in Lemma 1.17.2. Let us recall the notation there. We have an etale map $P \to \mathbf{A}^n_{O_K} = \text{Spec } O_K[T_1, \ldots, T_n]$. We use the notation T_i also for its pull-back to P. The closed immersion $U \to P$ is defined by the equation $\pi - u\prod_{i=1}^r T_i^{l_i}$ for a unit u on P, a prime element π of K, and integers $r \geq 0$ and $l_i \geq 1$. Let t_i denote the restriction of T_i on U. The divisor of U defined by the equation $t_i = 0$ is an irreducible component of the divisor $U_{s,\text{red}}$ with simple normal crossings. We consider P as a log scheme with respect to the standard log structure associated to the divisor $\Delta = (T_1 \cdot \cdots \cdot T_r = 0)$ with simple normal crossings. The immersion $U \to P$ is an exact immersion. Both of the map $\mathbf{N}^r \to \Gamma(P, O_P)$ sending the standard basis to T_i and the composition $\theta_P : M \to \Gamma(U, M_X) = \mathbf{N}^r \to \Gamma(P, O_P)$ induce charts of P. We define the log product $(P \times_S X)^\sim$ of the log schemes P and X over the scheme S with the *trivial* log structure with respect to the map $1 \to M$ of frames both for P and X.

Proposition 3.15 *Let the notation be as above.*

1. The second projection $pr_2 : (P \times_S X)^\sim \to X$ is classically smooth.

2. The map $(U \times_S X)^\sim \to (P \times_S X)^\sim$ induced by the immersion $U \to P$ is a regular immersion of codimension 1.

Proof. 1. Since $P \to S$ is log smooth, the projection $pr_2 : (P \times_S X)^\sim \to X$ is also log smooth. Since pr_2 is strict by Corollary 3.9, it is classically smooth.

2. By 1, the scheme $(P \times_S X)^\sim$ is regular. Hence it is sufficient to show that $(U \times_S X)^\sim$ is a subscheme of $(P \times_S X)^\sim$ locally defined by one equation. Choosing a numbering of the irreducible components of $X_{s,\mathrm{red}}$, we identify $M = \mathbf{N}^m$. The canonical map $N = \mathbf{N} \to M = \mathbf{N}^m$ maps 1 to $(l_1, \ldots, l_m)$ where l_i is the multiplicity of the divisor D_i in X_s. We take an open subscheme V of X where the frame $M \to \Gamma(X, \bar{M}_X)$ is lifted to a chart $\theta_V : M \to \Gamma(V, M_X)$. Let $s_i \in \Gamma(V, O_X)$ be the image of the standard basis. We show that $(U \times_S V)^\sim$ is a subscheme of $(P \times_S V)^\sim$ defined by one equation to complete the proof.

We compute $(U \times_S V)^\sim$ and $(P \times_S V)^\sim$ explicitly, using Corollary 3.10. Let $N \to \Gamma(S, M_S)$ be the chart sending 1 to a uniformizer π of K. It is a lifting of the standard frame $N \to \Gamma(S, \bar{M}_S)$. We extend the chart $\theta_P : M \to \Gamma(P, M_P)$ to a chart $\theta_P^\sim : M^\sim = M + \mathbf{Z} \to \Gamma(P, M_P)$ by sending $(0,1) \in M^\sim = M + \mathbf{Z}$ to $u = \pi / \prod_{i=1}^r T_i^{l_i} \in \Gamma(P, O_P^\times)$. We also extend the chart θ_V to a chart $\theta_V^\sim : M^\sim = M + \mathbf{Z} \to \Gamma(V, M_X)$ by sending $(0,1)$ to $v = \pi / \prod_{i=1}^m s_i^{l_i} \in \Gamma(V, O_V^\times)$. We define a map $\varphi : N \to M^\sim = M + \mathbf{Z}$ of monoids by sending 1 to $(l_1, \ldots, l_m, 1)$. It is a map of charts with respect to both of U and V. By Corollary 3.10, we have isomorphisms

$$\begin{aligned}(U \times_S V)^\sim &= (U \times_S V) \otimes_{\mathbf{Z}[M^\sim + M^\sim]} \mathbf{Z}[M^\sim + (M^{\sim \mathrm{gp}}/\varphi N^{\mathrm{gp}})] \\ (P \times_S V)^\sim &= (P \times_S V) \otimes_{\mathbf{Z}[M^\sim + M^\sim]} \mathbf{Z}[M^\sim + M^{\sim \mathrm{gp}}]\end{aligned}$$

where the map $s : M^\sim + M^\sim \to M^\sim + M^{\sim \mathrm{gp}}$ is given by $(a, b) \mapsto (a + b, a)$. Hence, we have

$$(U \times_S V)^\sim = U \times_P (P \times_S V)^\sim \otimes_{\mathbf{Z}[M^\sim + M^{\sim \mathrm{gp}}]} \mathbf{Z}[M^\sim + (M^{\sim \mathrm{gp}}/\varphi N^{\mathrm{gp}})].$$

Let w be the image of $(0, \varphi(1)) \in M^\sim + M^{\sim \mathrm{gp}}$ by the canonical map $M^\sim + M^{\sim \mathrm{gp}} \to \mathbf{Z}[M^\sim + M^{\sim \mathrm{gp}}] \to \Gamma((P \times_S V)^\sim, M_{(P \times_S V)^\sim})$. Then $(U \times_S V)^\sim$ is the closed subscheme of $(P \times_S V)^\sim$ defined by the ideal $I = ((\pi - u \prod_i T_i^{l_i}) \otimes 1, 1 - w)$. The map $s : M^\sim + M^\sim \to M^\sim + M^{\sim \mathrm{gp}}$ sends $(\varphi(1), 0)$ to $(\varphi(1), \varphi(1))$ and $(0, \varphi(1))$ to $(\varphi(1), 0)$. Since the images of $(\varphi(1), 0)$ and $(0, \varphi(1)) \in M^\sim + M^\sim$ in $\Gamma((P \times_S V)^\sim, O_{(P \times_S V)^\sim})$ are $(u \prod_i T_i^{l_i}) \otimes 1$ and $1 \otimes (v \prod_i s_i^{l_i}) = \pi$ respectively, we obtain $(u \prod_i T_i^{l_i}) \otimes 1 = w\pi$ and $(\pi - u \prod_i T_i^{l_i}) \otimes 1 = \pi(1 - w)$. Thus the ideal I is generated by $1 - w$ and the subscheme $(U \times_S V)^\sim$ is defined by one equation $1 - w$ in $(P \times_S V)^\sim$.

We apply the construction in Section 2 to define the logarithmic localized intersection product $[[\ , X]]_{(X \times_S X)^\sim} : G((X \times_S X)^\sim) \to G(X_s)$. By Proposition 3.15, the log product $(X \times_S X)^\sim$ is locally a hypersurface of relative dimension $n-1$ over X. Let $\tilde{i} : \tilde{Z} \to (X \times_S X)^\sim$ be the closed immersion defined by the ideal Ann $\Omega^n_{(X\times_S X)^\sim/X}$ and $\mathcal{L}_{\tilde{Z}}$ be the invertible $O_{\tilde{Z}}$-module $L^1\tilde{i}^*\Omega^1_{(X\times_S X)^\sim/X}$. Similarly, let $i : Z \to X$ be the closed immersion defined by the ideal Ann $\Lambda^n\Omega^1_{X/S}(\log/\log)$ and $\mathcal{L}_Z$ be the invertible O_Z-module $L^1 i^*\Omega^1_{X/S}(\log/\log)$.

Lemma 3.16 *Let $\Delta : X \to (X \times_S X)^\sim$ be the log diagonal map. Then, there exists a canonical isomorphism $L\Delta^*\Omega^1_{(X\times_S X)^\sim/X} \to \Omega^1_{X/S}(\log/\log)$. We have $Z = \tilde{Z} \times_{(X\times_S X)^\sim} X$ and a canonical isomorphism $\mathcal{L}_{\tilde{Z}}|_Z \to \mathcal{L}_Z$.*

Proof. Since the log diagonal map $\Delta : X \to (X\times_S X)^\sim$ is a section of the projection $(X\times_S X)^\sim \to X$, the pull-back $L\Delta^*\Omega^1_{(X\times_S X)^\sim/X}$ is equal to the conormal complex $M_{X/(X\times_S X)^\sim}$. Since X is regular and the immersion $X_K \to X_K \times_K X_K$ is regular, the complex $M_{X/(X\times_S X)^\sim}$ is quasi-isomorphic to the conormal sheaf $N_{X/(X\times_S X)^\sim}$ by Lemma 2.15.2. Hence the isomorphism $L\Delta^*\Omega^1_{(X\times_S X)^\sim/X} \to \Omega^1_{X/S}(\log/\log)$ follows from Proposition 3.13. The rest is clear.

Corollary 3.17 *The localized intersection product with log diagonal $[[\ ,X]]_{(X\times_S X)^\sim} : G((X \times_S X)^\sim) \to G(Z)_{/\mathcal{L}_{\tilde{Z}}}$ induces a map $[[\ ,X]] : G((X \times_S X)^\sim) \to G(Z) \to G(X_s)$.*

Proof. By Lemmas 3.16 and 1.26, we have $G(Z)_{/\mathcal{L}_{\tilde{Z}}} = G(Z)_{/\mathcal{L}_Z} = G(Z)$ similarly as in Example 2 after Definition 2.13 in §2.2. By the assumption that the generic fiber is smooth, the closed subset Z is a subset of the closed fiber X_s.

Definition 3.18 *We call the composite map*

$$[[\ ,X]]_{(X\times_S X)^\sim} : G((X\times_S X)^\sim) \longrightarrow G(Z) \longrightarrow G(X_s)$$

the logarithmic localized intersection product.

If there is no fear of confusion, we drop the suffix $_{(X\times_S X)^\sim}$. The map $[[X,\ ]]_{(X\times_S X)^\sim} : G((X \times_S X)^\sim) \to G(X_s)$ is the same as the map $[[\ ,X]]$.

Corollary 3.19 *1. The logarithmic self-intersection product $[[X,X]]_{(X\times_S X)^\sim} \in F_0G(X_s)$ is equal to the image of the logarithmic self-intersection cycle $(\Delta_X,\Delta_X)^{\log}_S \in CH_0(X_s)$:*

$$[[X,X]]_{(X\times_S X)^\sim} = (\Delta_X,\Delta_X)^{\log}_S.$$

2. Let n be the dimension of X. Then the map $[[\ ,X]] : G((X \times_S X)^\sim) \to G(X_s)$ sends the topological filtration $F_pG((X\times_S X)^\sim)$ into $F_{p-n}G(X_s)$.

Proof. 1. By Corollary 2.18.1 applied to the log diagonal map $X \to (X\times_S X)^\sim$, we have $[[X,X]]_{(X\times_S X)^\sim} = (-1)^n c_n{}^X_Z(M_{X/(X\times_S X)^\sim}) \cap [X]$ in $F_0G(X_s)$. In the proof of Lemma 3.16.1, we have shown that the conormal complex $M_{X/(X\times_S X)^\sim}$ is quasi-isomorphic to $\Omega^1_{X/S}(\log/\log)$. Thus we get $(-1)^n c_n{}^X_{X_s}(M_{X/(X\times_S X)^\sim}) \cap [X] = (-1)^n c_n{}^X_{X_s}(\Omega^1_{X/S}(\log/\log)) \cap [X] = (\Delta_X,\Delta_X)^{\log}_S$.

2. It suffices to apply Proposition 2.16.1 to the map $[[X,\ ]]_{(X\times_S X)^\sim} : G((X\times_S X)^\sim) \to G(Z)$.

The advantage of the logarithmic localized intersection product against the non-logarithmic one is the following Proposition. It claims that the logarithmic localized intersection product is factored through the generic fiber. The non-logarithmic product does not share this property in general.

Proposition 3.20 *Let K be a discrete valuation field with perfect residue field and X be a regular flat and separated scheme over $S = \text{Spec } O_K$ of finite type such that the generic fiber X_K is smooth and the reduced closed fiber has simple normal crossings. Then the map $[[\ ,X]]_{(X\times_S X)^\sim}$: $G((X\times_S X)^\sim)\to G(X_s)$ is factored by the surjection $G((X\times_S X)^\sim)\to G(X_K\times_K X_K)$.*

Corollary 3.21 *Let $d = n-1$ be the dimension of X_K. Then the induced map $[[\ ,X]] : G(X_K\times_K X_K)\to G(X_s)$ sends the topological filtration $F_pG(X_K\times_K X_K)$ into $F_{p-d}G(X_s)$.*

Proof of Corollary. Clear from Corollary 3.19.

The induced map $Gr^F_pG(X_K\times_K X_K)\to Gr^F_{p-d}G(X_s)$ is also denoted by $[[\ ,X]]$.

Proof of Proposition 3.20. Let $D_1,\dots,D_m$ be the irreducible components of X_s with the reduced subscheme structure. Let E_i be the fiber product $(X\times_S X)^\sim\times_X D_i$ with respect to the second projection. Since the open subscheme $X_K\times_K X_K$ of $(X\times_S X)^\sim$ is the complement of $\bigcup_{i=1}^m E_i$, we have an exact sequence

$$\bigoplus_{i=1}^m G(E_i)\longrightarrow G((X\times_S X)^\sim)\longrightarrow G(X_K\times_K X_K)\longrightarrow 0.$$

Hence it is sufficient to show that the composition

$$G(E_i)\longrightarrow G((X\times_S X)^\sim)\xrightarrow{[[\ ,X]]} G(X_s)$$

is the 0-map for each i. It follows from the two Lemmas below.

Lemma 3.22 *For each i, let l_i be the multiplicity of D_i in X_s and μ_{l_i,D_i} be the scheme of l_i-th roots of unity over D_i. Then, there exists a regular closed immersion $j_i : \mu_{l_i,D_i}\to E_i$ making the diagram*

$$\begin{array}{ccccc}
G(E_i) & \longrightarrow & G((X\times_S X)^\sim) & \xrightarrow{[[\ ,X]]} & G(X_s)\\
\| & & & & \uparrow\\
G(E_i) & \xrightarrow[j_i^*]{} & G(\mu_{l_i,D_i}) & \xrightarrow[{[[\ ,D_i]]_{\mu_{l_i,D_i}}}]{} & G(D_i).
\end{array}$$

Here the unlabeled arrows are the push-forward for closed immersions and D_i is regarded as a closed subscheme of μ_{l_i,D_i} by the unit section.

Lemma 3.23 *Let S be a regular noetherian scheme and $l\geq 1$ be an integer. We regard S as a closed subscheme of $X = \mu_{l,S}$ by the unit section $i : S\to X$. Then, the localized intersection product $[[\ ,S]]_X : G(X)\to G(S)$ is the 0-map.*

Proof of Lemma 3.23. The map $[[\ ,S]]_X : G(X)\to G(S)$ is well-defined as is remarked in Example 1 after Definition 2.13. First, we show that the map $[[\ ,S]]_X : G(X)\to G(S)$ is equal to the composition $G(X)\to G(\mathbf{G}_{m,S})\xrightarrow{i^*} G(S)$ where $i : S\to \mathbf{G}_{m,S}$ is the unit section. It is sufficient to apply Proposition 2.14 by taking $S = V = S, X = W = X$ and $P = \mathbf{G}_{m,S}$.

We show that the composition $G(X)\to G(\mathbf{G}_{m,S})\to G(S)$ is the 0-map. Let t be the coordinate of $\mathbf{G}_{m,S}$. Let $\mathcal{F}$ be a coherent O_X-module. Since $0\to O_{\mathbf{G}_{m,S}}\overset{(t-1)\times}{\to} O_{\mathbf{G}_{m,S}}\to O_S\to 0$ is a resolution of O_S by free $O_{\mathbf{G}_{m,S}}$-modules, we have a quasi-isomorphism $[\mathcal{F} \stackrel{(t-1)\times}{\to} \mathcal{F}] \to \mathcal{F} \otimes^L_{O_{\mathbf{G}_{m,S}}} O_S$. Hence the class $i^*[\mathcal{F}] \in G(S)$ is equal to the image of $0 = [\mathcal{F}] - [\mathcal{F}] \in G(X)$ by the push-forward map $G(X) \to G(S)$. Thus the assertion follows.

To prove Lemma 3.22, we introduce two self-log-products of an irreducible component D_i of the closed fiber X_s. An irreducible component D_i is smooth of dimension $n-1$ over the residue field F. We consider two log structures on D_i. Let M_{D_i} be the log structure of D_i induced from the standard log structure of X and let M'_{D_i} be the log structure defined by the divisor $\bigcup_{j\neq i}(D_j \cap D_i)$ with simple normal crossings. The canonical map $M = \Gamma(X, \bar{M}_X) \to \Gamma(D_i, \bar{M}_{D_i})$ and the identity $M' = \Gamma(D_i, \bar{M}'_{D_i}) \to \Gamma(D_i, \bar{M}'_{D_i})$ are frames. There is a canonical map $(D_i, M_{D_i}) \to (D_i, M'_{D_i})$ of log schemes. Similarly, we consider two log structures on $s = \text{Spec } F$. Let M_s be the log structure of s induced from the standard log structure of $S = \text{Spec } O_K$ and let M'_s be the trivial log structure. Let $(D_i \times_s D_i)^\sim$ be the log product of (D_i, M'_{D_i}) over (s, M'_s) and let $(D_i \times_S D_i)^\sim$ be the log product of (D_i, M_{D_i}) over (s, M_s). The canonical map $(D_i, M_{D_i}) \to (D_i, M'_{D_i})$ induces a map $(D_i \times_S D_i)^\sim \to (D_i \times_s D_i)^\sim$.

We reduce Lemma 3.22 to the following Lemma 3.24.

Lemma 3.24 *1. The canonical map $(D_i \times_S D_i)^\sim \to (X \times_S X)^\sim$ induces an isomorphism $(D_i \times_S D_i)^\sim \to E_i = (X \times_S X)^\sim \times_X D_i$.*

2. Let l_i be the multiplicity of D_i in X_s. The scheme $(D_i \times_S D_i)^\sim$ is a μ_{l_i}-torsor over $(D_i \times_s D_i)^\sim$.

3. The (second) projection $(X \times_S X)^\sim \to X$ is flat.

Proof of Lemma 3.24 ⇒ 3.22. We identify $(D_i \times_S D_i)^\sim = E_i$, by the isomorphism in Lemma 3.24.1. The base change $(D_i \times_S D_i)^\sim \times_{(D_i \times_s D_i)^\sim} D_i$ by the log diagonal map $D_i \to (D_i \times_s D_i)^\sim$ is a μ_{l_i}-torsor over D_i by Lemma 3.24.2. The log diagonal map $D_i \to (D_i \times_S D_i)^\sim$ induces a section $D_i \to (D_i \times_S D_i)^\sim \times_{(D_i \times_s D_i)^\sim} D_i$. Hence it is isomorphic to μ_{l_i, D_i}. We identify $(D_i \times_S D_i)^\sim \times_{(D_i \times_s D_i)^\sim} D_i$ with μ_{l_i, D_i}. We show that the immersion

$$j_i : (D_i \times_S D_i)^\sim \times_{(D_i \times_s D_i)^\sim} D_i = \mu_{l_i, D_i} \to (D_i \times_S D_i)^\sim = E_i$$

is a regular immersion. Since the projection $(D_i \times_s D_i)^\sim \to D_i$ is log smooth and strict, it is smooth. Hence the log diagonal map $\Delta_{D_i} : D_i \to (D_i \times_s D_i)^\sim$ is a regular immersion. Since $(D_i \times_S D_i)^\sim$ is flat over $(D_i \times_s D_i)^\sim$ by Lemma 3.24.2, the immersion $(D_i \times_S D_i)^\sim \times_{(D_i \times_s D_i)^\sim} D_i \to (D_i \times_S D_i)^\sim$ is also a regular immersion.

We show that the diagram is commutative. We apply Corollary 2.24.2 by taking the log diagonal $X \to (X \times_S X)^\sim$ to be $V \to X$ in Corollary 2.24.2, X to be S and D_i to be T, to conclude the diagram

$$\begin{array}{ccc} G((X \times_S X)^\sim) & \xrightarrow{[[\ ,X]]} & G(X_s) \\ \uparrow & & \uparrow \\ G(E_i) & \xrightarrow[{[[\ ,D_i]]_{E_i}}]{} & G(D_i) \end{array}$$

is commutative. By Lemma 3.24.3, the assumption that X is flat over S in Corollary 2.24 is satisfied and the diagram is commutative. We show that the map $[[\ ,D_i]]_{E_i} : G(E_i) \to G(D_i)$ is equal to the composition of the lower line of the diagram in Lemma 3.22 by applying Corollary 2.25.3. We take D_i to be $S = S'$ in Corollary 2.25.3, E_i to be X, $(D_i \times_S D_i)^\sim \times_{(D_i \times_s D_i)^\sim} D_i$ to be $W = X'$ and the log diagonal $D_i \to (D_i \times_S D_i)^\sim \times_{(D_i \times_s D_i)^\sim} D_i$ to be $V' \to W$. Then, since the immersion $j_i : (D_i \times_S D_i)^\sim \times_{(D_i \times_s D_i)^\sim} D_i \to E_i$ is a regular immersion, the assumption is satisfied. Since $(D_i \times_S D_i)^\sim \times_{(D_i \times_s D_i)^\sim} D_i \simeq \mu_{l_i, D_i}$, the invertible module $\mathcal{L}'_{Z'}$ is trivial. Thus the assertion follows.

Proof of Lemma 3.24. 1. By the universality of log product, the canonical map $(D_i \times_S D_i)^\sim \to (X \times_S X)^\sim \times_{X \times_S X} (D_i \times_s D_i)$ is an isomorphism. Hence it is sufficient to show that the immersion $(X \times_S X)^\sim \times_{X \times_S X} (D_i \times_s D_i) \to E_i = (X \times_S X)^\sim \times_X D_i$ is an isomorphism. The question is local on $X \times_S X$. Let U and V be open subschemes of X and take local charts $M^\sim = M + \mathbf{Z} \to \Gamma(U, M_X)$ and $M^\sim \to \Gamma(V, M_X)$ lifting the standard frames. Let M' be the quotient monoid of $M = \Gamma(X, \bar{M}_X) \simeq \mathbf{N}^m$ by the i-th component $N_i \simeq \mathbf{N}$ and we put $M'^\sim = M' + \mathbf{Z}$. Then by Corollary 3.10, we have

$$\begin{aligned} & (U \times_S V)^\sim \times_{X \times_S X} (D_i \times_s D_i) \\ = {} & (U \times_S V) \otimes_{\mathbf{Z}[M^\sim + M^\sim]} \mathbf{Z}[M^\sim + M^{\sim\mathrm{gp}}/N^{\mathrm{gp}}] \otimes_{\mathbf{Z}[M^\sim + M^\sim]} \mathbf{Z}[M'^\sim + M'^\sim], \\ & (U \times_S V)^\sim \times_X D_i \\ = {} & (U \times_S V) \otimes_{\mathbf{Z}[M^\sim + M^\sim]} \mathbf{Z}[M^\sim + M^{\sim\mathrm{gp}}/N^{\mathrm{gp}}] \otimes_{\mathbf{Z}[M^\sim]} \mathbf{Z}[M'^\sim]. \end{aligned}$$

In the tensor products, the map $M^\sim + M^\sim \to M^\sim + M^{\sim\mathrm{gp}}/N^{\mathrm{gp}}$ is $(a, b) \mapsto (a + b, a)$ and $M^\sim \to M^\sim + M^{\sim\mathrm{gp}}/N^{\mathrm{gp}}$ is $a \mapsto (a, a)$. Both of the tensor products $\mathbf{Z}[M^\sim + M^{\sim\mathrm{gp}}] \otimes_{\mathbf{Z}[M^\sim + M^\sim]} \mathbf{Z}[M'^\sim + M'^\sim]$ and $\mathbf{Z}[M^\sim + M^{\sim\mathrm{gp}}] \otimes_{\mathbf{Z}[M^\sim]} \mathbf{Z}[M'^\sim]$ are naturally identified with $\mathbf{Z}[M'^\sim + M^{\sim\mathrm{gp}}]$. Thus the assertion follows.

2. The question is local on $X \times_S X$. Let U and V be open subschemes of X and take local charts $M \to \Gamma(U, M_X)$ and $M \to \Gamma(V, M_X)$ lifting the standard frame $\mathrm{id} : M = \Gamma(X, \bar{M}_X) \to \Gamma(X, \bar{M}_X)$. We put $M^\sim = M \oplus \mathbf{Z}$ and define charts $M^\sim \to \Gamma(U, M_X)$ and $M^\sim \to \Gamma(V, M_X)$ extending the charts $M \to \Gamma(U, M_X)$ and $M \to \Gamma(V, M_X)$ and a morphism of charts $N \to M^\sim$ both for U and V as in Corollary 3.7.3. They are described explicitly as follows.

Let $D_1, \ldots, D_m$ be the irreducible components of X_s and identify $M = \mathbf{N}^m, N = \mathbf{N}$. Let l_i be the multiplicity of D_i in the closed fiber X_s. The canonical map $N \to M$ send 1 to $l = (l_1, \ldots, l_m) \in M$. Let $t_j \in \Gamma(U, O_U)$ be the image of the standard basis $e_j \in M$ for $j = 1, \ldots, m$. Then the chart $M^\sim = M + \mathbf{Z} \to \Gamma(U, M_X)$ is defined by $(0, 1) \mapsto u = \pi / \prod_{j=1}^m t_j^{l_j} \in \Gamma(U, O_X^\times)$. The chart $M^\sim \to \Gamma(V, M_X)$ is defined similarly by sending $(0, 1)$ to $v = \pi / \prod_{j=1}^m s_j^{l_j} \in \Gamma(V, O_X^\times)$. The map $M^\sim \to M$ is the projection and the map $N \to M^\sim$ is defined by $1 \mapsto (l, 1)$.

We also define local charts lifting the standard frame $\mathrm{id} : M' = \Gamma(D_i, \bar{M}'_{D_i}) \to \Gamma(D_i, \bar{M}'_{D_i})$. We identify $M' \simeq \bigoplus_{j=1,\ldots,m, \neq i} \mathbf{N}$. Let $U_i = U \cap D_i$ and $V_i = V \cap D_i$. The map $M' \to \Gamma(U_i, M'_{D_i})$ sending the standard basis e_j to t_j for $j \neq i$ is a chart lifting the standard frame $M' \to \Gamma(D_i, \bar{M}'_{D_i})$. Similarly, we define a chart $M' \to \Gamma(V_i, M'_{D_i})$ lifting the frame $M' \to \Gamma(D_i, \bar{M}'_{D_i})$. The compositions $M^\sim \to \Gamma(U, M_X) \to \Gamma(U_i, M_{D_i})$ and $M^\sim \to \Gamma(V_i, M_{D_i})$ are local charts lifting the induced frame $M = \Gamma(X, \bar{M}_X) \to \Gamma(D_i, \bar{M}_{D_i})$.

By Corollary 3.10, we have canonical isomorphisms

$$\begin{aligned}(U_i \times_S V_i)^\sim &= (U_i \times_s V_i) \otimes_{\mathbf{Z}[M^\sim + M^\sim]} \mathbf{Z}[M^\sim + M^{\sim\mathrm{gp}}/N^{\mathrm{gp}}]\\ (U_i \times_s V_i)^\sim &= (U_i \times_s V_i) \otimes_{\mathbf{Z}[M' + M']} \mathbf{Z}[M' + M'^{\mathrm{gp}}].\end{aligned}$$

We put $M'^\sim = M' + \mathbf{Z}$ and regard it as a quotient of $M^\sim = M'^\sim + N_i$. Note that the maps $M^\sim \to \Gamma(U_i, O_{D_i})$ and $M^\sim \to \Gamma(V_i, O_{D_i})$ are factored by $M^\sim \to M'^\sim$. By an elementary computation, we see that the canonical map of monoids induce isomorphisms

$$\begin{aligned}&\mathbf{Z}[M'^\sim + M'^\sim] \otimes_{\mathbf{Z}[M^\sim + M^\sim]} \mathbf{Z}[M^\sim + M^{\sim\mathrm{gp}}] \to \mathbf{Z}[M'^\sim + M^{\sim\mathrm{gp}}]\\ \leftarrow\ &\mathbf{Z}[M'^\sim + M'^\sim] \otimes_{\mathbf{Z}[M' + M']} \mathbf{Z}[M' + M'^{\mathrm{gp}}] \otimes_{\mathbf{Z}} \mathbf{Z}[N_i^{\mathrm{gp}}].\end{aligned}$$

In the tensor product, the maps $\mathbf{Z}[M^\sim + M^\sim] \to \mathbf{Z}[M^\sim + M^{\sim\mathrm{gp}}]$, $\mathbf{Z}[M' + M'] \to \mathbf{Z}[M' + M'^{\mathrm{gp}}]$ are induced by the map $(a, b) \mapsto (a + b, a)$ of monoids. We put $l' = (l_1, \ldots, \check{l}_i, \ldots, l_m) \in M'$. Then the image of $1 \in N = \mathbf{N}$ by the canonical map $N \to M^\sim$ is $((l', 1), l_i) \in M^\sim = M'^\sim + N_i$. By the isomorphism above, the element $1 \otimes (0, (l, 1)) \in \mathbf{Z}[M'^\sim + M'^\sim] \otimes_{\mathbf{Z}[M^\sim + M^\sim]} \mathbf{Z}[M^\sim + M^{\sim\mathrm{gp}}]$ is sent to $1 \otimes (0, (l', 1)) \otimes (l_i) \in \mathbf{Z}[M'^\sim + M'^\sim] \otimes_{\mathbf{Z}[M' + M']} \mathbf{Z}[M' + M'^{\mathrm{gp}}] \otimes_{\mathbf{Z}} \mathbf{Z}[N_i^{\mathrm{gp}}]$. Hence we have an isomorphism

$$\begin{aligned}&\mathbf{Z}[M'^\sim + M'^\sim] \otimes_{\mathbf{Z}[M^\sim + M^\sim]} \mathbf{Z}[M^\sim + M^{\sim\mathrm{gp}}/N^{\mathrm{gp}}]\\ \to\ &\mathbf{Z}[M'^\sim + M'^\sim] \otimes_{\mathbf{Z}[M' + M']} \mathbf{Z}[M' + M'^{\mathrm{gp}}] \otimes_{\mathbf{Z}} \mathbf{Z}[N_i^{\mathrm{gp}}]/(1 \otimes (0, (l', 1)) \otimes (l_i) - 1).\end{aligned}$$

Let $\alpha = \frac{u \otimes 1}{1 \otimes v} \prod_{j \neq i} (\frac{t_j \otimes 1}{1 \otimes s_j})^{l_j} \in \Gamma((U_i \times_s V_i)^\sim, O^\times_{(D_i \times_s D_i)^\sim})$ be the image of $(0, (l', 1)) \in M' + M'^{\mathrm{gp}}$ and T be the image of $1 \in N_i^{\mathrm{gp}}$ in $\mathbf{Z}[N_i^{\mathrm{gp}}] = \mathbf{Z}[T^{\pm 1}]$. Then we obtain

$$\begin{aligned}(U_i \times_S V_i)^\sim &= (U_i \times_s V_i)^\sim \otimes_{\mathbf{Z}} \mathbf{Z}[N_i^{\mathrm{gp}}]/(1 \otimes (0, (l', 1)) \otimes (l_i) - 1)\\ &= (U_i \times_s V_i)^\sim \otimes_{\mathbf{Z}} \mathbf{Z}[T^{\pm 1}]/(\alpha T^{l_i} - 1).\end{aligned}$$

We define an action of μ_{l_i} on $(U_i \times_S V_i)^\sim$ by the multiplication on T. Then it is easily checked that the action is well-defined on $(D_i \times_S D_i)^\sim$. Thus the assertion is proved.

3. The question is local on $X \times_S X$. Let U be an open subscheme of X and $U \to P$ be as in Proposition 3.15. It is sufficient to show that the divisor $(U \times_S X)^\sim$ of $(P \times_S X)^\sim$ is a relative divisor over X. The condition is checked fiberwise. On the generic fiber, it follows from the assumption that the generic fiber X_K is smooth. On the closed fiber, it is sufficient to show that $(X \times_S X)^\sim \times_X D_i = (D_i \times_S D_i)^\sim$ is flat of relative dimension $n - 1$ over D_i for each irreducible component D_i. Since $(D_i \times_S D_i)^\sim$ is finite flat over $(D_i \times_s D_i)^\sim$ and $(D_i \times_s D_i)^\sim$ is smooth of relative dimension $n - 1$ over D_i, the assertion follows.

3.4. Correspondences.

We formulate a generalization of Theorem 1.15 for an algebraic correspondence. To state it, we prepare some terminology and notations on the cycle map and algebraic correspondences. As in the last subsection, K is a discrete valuation field with perfect residue field F, $S = \operatorname{Spec} O_K$ and $s = \operatorname{Spec} F$ is the closed point of S. Let X_K be a proper smooth scheme over K and ℓ be a prime number different from the characteristic of the residue field F. Then, for an integer $r \geq 0$, we have a cycle map $cl : CH^r(X_K) \to H^{2r}(X_{\bar{K}}, \mathbf{Q}_\ell(r))$. For $\Gamma \in CH^r(X_K)$, the image $cl(\Gamma)$ is also denoted by $[\Gamma]$. It is compatible with the product and the pull-back. It also makes the degree map $\deg : CH_0(X_K) \to \mathbf{Z}$ compatible with the trace map. Its composition with the chern character map $ch : Gr^r_F K(X_K) \to CH^r(X_K)_{\mathbf{Q}}$ is the chern character map $ch : Gr^r_F K(X_K) \to H^{2r}(X_{\bar{K}}, \mathbf{Q}_\ell(r))$. Let Y_K be another proper smooth schemes over K and assume $\dim X_K = \dim Y_K = d$. We call an element $\Gamma \in CH_d(X_K \times_K Y_K)$ an algebraic correspondence from X_K to Y_K. An algebraic correspondence $\Gamma \in CH_d(X_K \times_K Y_K)$ defines a G_K-equivariant map $\Gamma^* : H^*(Y_{\bar{K}}, \mathbf{Q}_\ell) \to H^*(X_{\bar{K}}, \mathbf{Q}_\ell)$ as the composition

$$H^*(Y_{\bar{K}}, \mathbf{Q}_\ell) \xrightarrow{pr_2^*} H^*(X_{\bar{K}} \times_{\bar{K}} Y_{\bar{K}}, \mathbf{Q}_\ell) \xrightarrow{[\Gamma]\cup} H^*(X_{\bar{K}} \times_{\bar{K}} Y_{\bar{K}}, \mathbf{Q}_\ell) \xrightarrow{pr_{1*}} H^*(X_{\bar{K}}, \mathbf{Q}_\ell).$$

When $X_K = Y_K$, an algebraic correspondence Γ on X_K defines an endomorphism Γ^* of the ℓ-adic representation $H^q(X_{\bar{K}}, \mathbf{Q}_\ell)$ of G_K. We put

$$\mathrm{Sw}(\Gamma, X_K/K) = \sum_{q=0}^{2d} (-1)^q \mathrm{Sw}(\Gamma^* : H^q(X_{\bar{K}}, \mathbf{Q}_\ell)).$$

For an endomorphism $f : X_K \to X_K$ over K, similarly we put

$$\mathrm{Sw}(f, X_K/K) = \sum_{q=0}^{2\dim X_K} (-1)^q \mathrm{Sw}(f^* : H^q(X_{\bar{K}}, \mathbf{Q}_\ell)).$$

If $\Gamma_f \in CH_d(X_K \times_K X_K)$ is the class of the graph of f, we have $\mathrm{Sw}(f, X_K/K) = \mathrm{Sw}(\Gamma_f, X_K/K)$. If $f = \mathrm{id}$, we have $\mathrm{Sw}(\mathrm{id}, X_K/K) = \mathrm{Sw}(X_K/K)$.

Let X be a proper and flat regular scheme over $S = \mathrm{Spec}\, O_K$ such that $X \otimes_{O_K} K = X_K$ and that the reduced closed fiber $X_{s,\mathrm{red}}$ has simple normal crossings. For $\Gamma \in CH_d(X_K \times_K X_K)$, let $[[\Gamma, X]] \in Gr^F_0 G(X_s)$ be the image by the composition map $CH_d(X_K \times_K X_K) \to Gr^F_d G(X_K \times_K X_K) \stackrel{[[\ ,X]]}{\to} Gr^F_0 G(X_s)$. We define the degree map $\deg_{X_s} : G(X_s) \to G(s) = \mathbf{Z}$ to be the push-forward for $X_s \to s$.

Theorem 3.25 *Let K be a discrete valuation field with perfect residue field and ℓ be a prime number different from the characteristic of the residue field. Let X_K be a proper smooth scheme over K of dimension d. Let $\Gamma \in CH_d(X_K \times_K X_K)$ be an algebraic correspondence on X_K. Then,*

1. $\mathrm{Sw}(\Gamma, X_K/K)$ is a rational number independent of ℓ.

2. Let X be a proper and flat regular scheme over $S = \mathrm{Spec} O_K$ such that $X \otimes_{O_K} K = X_K$ and that the reduced closed fiber $X_{s,\mathrm{red}}$ has simple normal crossings. Then we have an equality of integers

$$\mathrm{Sw}(\Gamma, X_K/K) = -\deg_{X_s} [[\Gamma, X]].$$

Proof will be given in Section 4. Theorem 1.15, which is shown to be equivalent to Theorem 1.10, is the special case of the following Corollary where $f = \mathrm{id}$, by Corollary 3.19.

Corollary 3.26 *Let K, X_K and ℓ be as in Theorem 3.25. Let $f : X_K \to X_K$ be an endomorphism over K. Then,*

1. $\mathrm{Sw}(f, X_K/K)$ *is a rational number independent of ℓ.*

2. Let X be a proper and flat regular scheme over $S = \mathrm{Spec} O_K$ such that $X \otimes_{O_K} K = X_K$ and that the reduced closed fiber $X_{s,\mathrm{red}}$ has simple normal crossings. Let $\Gamma_f \in CH_d(X_K \times_K X_K)$ be the class of the graph of f. Then we have an equality of integers

$$\mathrm{Sw}(f, X_K/K) = -\deg_{X_s}[[\Gamma_f, X]].$$

Proof of Corollary. It is enough to apply Theorem 3.25 to Γ_f.

Let $X \to S$ be as in Theorem 3.25.2. We say an automorphism σ of X over S is admissible if the following condition is satisfied.

For each irreducible component D_i of the reduced closed fiber $X_{s,\mathrm{red}}$, we have either $\sigma(D_i) = D_i$ or $\sigma(D_i) \cap D_i = \emptyset$.

For an admissible automorphism σ of X over S, the localized intersection product $[[\Gamma_\sigma, X]]$ is computed using the Segre classes as follows.

Lemma 3.27 *Let σ be an admissible automorphism of X over S and $n = \dim X$. We put $U = X - \bigcup_{i:\sigma(D_i)\cap D_i=\emptyset} D_i$. Then,*

1. The pair $(1, \sigma) : U \to X$ of maps induces a closed immersion $U \to (X \times_S X)^\sim$.

2. Let Γ_σ denote U regarded as a closed subscheme of $(X \times_S X)^\sim$ by the immersion in 1 and let $\Delta_U \subset (U \times_S U)^\sim$ denote the log diagonal. Then the intersection $X^\sigma_{\log} = X \times_{(X\times_S X)^\sim} \Gamma_\sigma$ is equal to $\Delta_U \times_{(U\times_S U)^\sim} \Gamma_\sigma$.

3. Assume that σ does not have a fixed point in the generic fiber X_K. Then the localized intersection product $[[\Gamma_\sigma, X]]_{(X\times_S X)^\sim} \in Gr^F_0 G(X^\sigma_{\log})$ is equal to the image of

$$\{c(\Omega^1_{X/S}(\log/\log))^* \cap s(X^\sigma_{\log}, X)\}_{\dim 0} = \sum_{i=0}^{n-1} (-1)^i c_i(\Omega^1_{X/S}(\log/\log)) s_{n-i}(X^\sigma_{\log}, X).$$

In particular, if the closed subscheme $X^\sigma_{\log}$ is a Cartier divisor of X, we have

$$[[\Gamma_\sigma, X]]_{(X\times_S X)^\sim} = \{c(\Omega^1_{X/S}(\log/\log))^* \cap (1 + X^\sigma_{\log})^{-1} \cap [X^\sigma_{\log}]\}_{\dim 0}.$$

Proof. 1. We set $(X \times_S X)^0 = X \times_S X - \bigcup_{(i,j):D_i\cap D_j=\emptyset} D_i \times D_j$. We show that $(X \times_S X)^\sim$ is a scheme over $(X \times_S X)^0$. By the definition of $(X \times_S X)^\sim$, we have $pr_1^{-1}(D_i) = pr_2^{-1}(D_i)$ in $(X \times_S X)^\sim$. Hence the inverse image of $\bigcup_{(i,j):D_i\cap D_j=\emptyset} D_i \times D_j = X \times_S X - (X \times_S X)^0$ in $(X \times_S X)^\sim$ is $\bigcup_{(i,j):D_i\cap D_j=\emptyset} pr_1^{-1}(D_i) \cap pr_2^{-1}(D_j) = \bigcup_{(i,j):D_i\cap D_j=\emptyset} pr_2^{-1}(D_i \cap D_j) = \emptyset$. Thus the claim is proved. By the definition of U, it is the inverse image of $(X \times_S X)^0 \subset X \times_S X$ by the map $(1, \sigma) : X \to X \times_S X$. Hence the map $U \to (X \times_S X)^0$ is a closed immersion. By the admissibility condition, the map $(1, \sigma) : X \to X \times_S X$ induces a map $U \to (X \times_S X)^\sim$. Since $U \to (X \times_S X)^0$ is a closed immersion, the induced map $U \to (X \times_S X)^\sim$ is also a closed immersion.

2. Since U is stable under σ, Γ_σ is a subscheme of $(U \times_S U)^\sim \subset (X \times_S X)^\sim$. The assertion follows from $\Delta_U = X \times_{(X\times_S X)^\sim} (U \times_S U)^\sim$.

3. By the assumption that σ does not have a fixed point in the generic fiber X_K, the underlying set of $X^\sigma_{\log}$ is a subset of the closed fiber X_s. We apply Corollary 2.21, by taking $X \to (X\times_S X)^\sim \to X$ to be $V \to X \to S$ in Corollary 2.21 and $X^\sigma_{\log} \to \Gamma_\sigma \to (X \times_S X)^\sim$ to be $T \to W \to X$. Since $M_{X/(X\times_S X)^\sim} = \Omega^1_{X/S}(\log/\log)$, we obtain $[[X, \Gamma_\sigma]]_{(X\times_S X)^\sim} = \{c(\Omega^1_{X/S}(\log/\log))^* \cap s(X^\sigma_{\log}, \Gamma_\sigma)\}_{\dim 0}$. By the automorphism $(x, y) \mapsto (y, \sigma(x))$ of $(U \times_S U)^\sim$, the closed subschemes Δ_U and Γ_σ are interchanged. Hence by 2, we have $s(X^\sigma_{\log}, \Gamma_\sigma) = s(X^\sigma_{\log}, \Delta_U) = s(X^\sigma_{\log}, X)$. Thus the assertion is proved.

The following corollary is an immediate consequence of Theorem 3.25.2.

Corollary 3.28 *Let K, X and ℓ be as in Theorem 3.25.2. Let $\Gamma \in CH_d(X_K \times_K X_K) \otimes \mathbf{Q}_\ell$ and assume that there exist an integer q_0 and a subspace $V \subset H^{q_0}(X_{\bar K}, \mathbf{Q}_\ell)$ such that the endomorphism Γ^* on $H^q(X_{\bar K}, \mathbf{Q}_\ell)$ is the projector to V if $q = q_0$ and is 0 otherwise. Then we have an equality of integers*

$$(-1)^{q_0}\mathrm{Sw}(V) = -\deg_{X_s}[[\Gamma, X]].$$

We show that Theorem 3.25 is reduced to the case where K is complete.

Lemma 3.29 *Let K be a discrete valuation field with perfect residue field and let X be a regular flat scheme of dimension n of finite type over O_K with smooth generic fiber. Assume that the reduced closed fiber $X_{s,\mathrm{red}}$ has simple normal crossings. Let K' be a discrete valuation field with perfect residue field. Assume that K' is an extension of K, the valuation of K' is an extension of that of K and that a prime element of K is a prime element of K'. Put $S' = \mathrm{Spec}\ O_{K'}$ and let s' be the closed point of S'. Then,*

1. *$X' = X \times_S S'$ is regular and the reduced closed fiber $X'_{s',\mathrm{red}}$ has simple normal crossings.*
2. *We have a commutative diagram*

$$\begin{array}{ccc} G((X\times_S X)^\sim) & \xrightarrow{[[\ ,X]]_{(X\times_S X)^\sim}} & G(X_s) \\ \downarrow & & \downarrow \\ G((X'\times_{S'} X')^\sim) & \xrightarrow[{[[\ ,X']]_{(X'\times_{S'} X')^\sim}}]{} & G(X'_{s'}) \end{array}$$

where the vertical arrows are the pull-backs.

Proof. The assertion 1 is checked easily using Lemma 1.17.2. We show 2. We have $(X' \times_{S'} X')^\sim = (X \times_S X)^\sim \times_S S'$ and the vertical arrows are defined. We show that the both compositions are equal to the map $[[\ , X']]_{(X\times_S X)^\sim} : G((X \times_S X)^\sim) \to G(X'_{s'})$ by applying Corollary 2.25. For the composition via $G(X_s)$, it suffices to apply Corollary 2.25.1 by taking $(X \times_S X)^\sim$ to be X in Corollary 2.25.1, X to be W and X' to be W'. For the composition via $G((X' \times_{S'} X')^\sim)$, we take $(X \times_S X)^\sim$ to be X in Corollary 2.25.3 and $(X' \times_{S'} X')^\sim$ to be $X' = W$. Then since $(X' \times_{S'} X')^\sim = (X \times_S X)^\sim \times_S S' \to (X \times_S X)^\sim$ is flat and hence of finite tor-dimension, the assumption in Corollary 2.25.3 is satisfied. Hence the assertion follows.

Corollary 3.30 *Let X, K and Γ be as in Theorem 3.25. Let $K' \supset K$ be a discrete valuation field with perfect residue field. Assume that the valuation of K' is an extension of that of K and a prime element of K is a prime element of K'. Then Theorem 3.25 for X and Γ is equivalent to that for $X' = X \otimes_{O_K} O_{K'}$ and the pull-back Γ' of Γ to $X'_{K'} \times_{K'} X'_{K'}$.*

Proof. As is remarked in §1.1, we have $\mathrm{Sw}(\Gamma, X_K/K) = \mathrm{Sw}(\Gamma', X_{K'}/K')$. By Lemma 3.29, we have $\deg_{X_s}[[\Gamma, X]] = \deg_{X'_{s'}}[[\Gamma', X']]$.

3.5. Some properties of log products.

We establish some properties of log products used in the proofs of Theorem 3.25 and of log Lefschetz trace formula.

Let K be a discrete valuation field and X and Y be regular flat schemes of finite type over $S = \mathrm{Spec}\ O_K$. We assume that the reduced closed fibers $X_{s,\mathrm{red}}$ and $Y_{s,\mathrm{red}}$ are divisors with simple normal crossings. We do not assume that the generic fibers are smooth. We regard X, Y and S as log schemes with respect to the standard log structures associated to the reduced closed fibers. Let $N = \mathbf{N} = \Gamma(S, \bar{M}_S)$ be the standard frame. If frames $M \to \Gamma(X, \bar{M}_X)$ and $M \to \Gamma(Y, \bar{M}_Y)$ and a morphism $N \to M$ of frames with respect to both of X and Y are given, the log product $(X \times_S Y)^\sim$ with respect to $N \to M$ is defined.

Proposition 3.31 *Let X and Y be regular and flat schemes over S of finite type such that the reduced closed fibers have simple normal crossings and let $f : X \to Y$ be a morphism over S. Then, the induced map $(f \times f)^\sim : (X \times_S X)^\sim \to (Y \times_S Y)^\sim$ of the log products with respect to the standard frames is of finite tor-dimension.*

To prove it, we show that we can take immersions as in Lemma 1.17.2 compatible with f.

Lemma 3.32 *Let the notation be as in Proposition 3.31. Let x be a point in the closed fiber of X and put $y = f(x)$. Then there exist open neighborhoods $U \ni x$ and $V \ni y$ satisfying $f(U) \subset V$, a morphism $g : P \to Q$ of smooth schemes over O_K and regular immersions $U \to P$ and $V \to Q$ of codimension 1 satisfying the following conditions (1) and (2).*

(1). Let $D_1, \ldots, D_r$ be the irreducible components of the closed fiber of U and let $D'_1, \ldots, D'_s$ be the irreducible components of the closed fiber of V and let e_{ij} be the integers such that $f^*D'_j = \sum_i e_{ij} D_i$. Then there are smooth divisors $E_1, \ldots, E_r$ of P and $E'_1, \ldots, E'_s$ of Q respectively such that $\bigcup_{i=1}^r E_i$ and $\bigcup_{j=1}^s E'_j$ are divisors with relative normal crossings, $D_i = U \times_P E_i$ for $i = 1, \ldots r$ and $D'_j = V \times_Q E'_j$ for $j = 1, \ldots s$ and we have $f^* E'_j = \sum_i e_{ij} E_i$.

(2). The diagrams

$$\begin{array}{ccc} U & \longrightarrow & P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ V & \longrightarrow & Q \end{array} \qquad \text{and} \qquad \begin{array}{ccc} (U \times_S X)^\sim & \longrightarrow & (P \times_S X)^\sim \\ \downarrow & & \downarrow \\ (V \times_S Y)^\sim & \longrightarrow & (Q \times_S Y)^\sim \end{array}$$

are cartesian.

Proof of Lemma 3.32. Let $t_1, \ldots, t_r$ be elements of $O_{X,x}$ defining the irreducible components D_i and let $t'_1, \ldots, t'_s$ be elements of $O_{Y,y}$ defining the irreducible components D'_j as in the proof of Lemma 1.17.2. Let $n = \dim X$ and $m = \dim Y$. We define maps $U \to P_1 \to \mathbf{A}^n_{O_K}$ and $V \to \mathbf{A}^m_{O_K}$ as in Lemma 1.17.2 using $t_1, \ldots, t_r$ and $t'_1, \ldots, t'_s$ respectively. We define a map $g_1 : P_1 \to \mathbf{A}^m_{O_K}$ making the diagram

$$\begin{array}{ccc} U & \longrightarrow & P_1 \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g_1} \\ V & \longrightarrow & \mathbf{A}^m_{O_K} \end{array}$$

commutative as follows. For each $1 \leq j \leq s$, take a lifting $u_j \in O^\times_{P,x}$ of $f^*t'_j / \prod_i t_i^{e_{ij}} \in O^\times_{X,x}$. For each $s+1 \leq j \leq m$, take a lifting $v_j \in O_{P,x}$ of $f^*t'_j \in O_{X,x}$. Shrinking U and P_1 if necessary, we define a map $g_1 : P_1 \to \mathbf{A}^m_{O_K} = \mathrm{Spec} O_K[T'_1, \ldots, T'_m]$ by $T'_j \mapsto u_j \prod_i T_i^{e_{ij}}$ for $1 \leq j \leq s$ and $T'_j \mapsto v_j$ for $s+1 \leq j \leq m$. Shrinking V if necessary, we take a immersion $V \to Q$ and an etale morphism $Q \to \mathbf{A}^m_{O_K}$. Putting $P = P_1 \times_{\mathbf{A}^m_{O_K}} Q$, we obtain the left commutative diagram in the condition (2).

We verify that the conditions are satisfied. The condition (1) is clear from the construction. We show that the first diagram in (2) is cartesian. Let e_i be the multiplicity of D_i in the closed fiber $X_s = \sum_i e_i D_i$. Let e'_j be the multiplicity of D'_j in the closed fiber $Y_s = \sum_j e'_j D'_j$. Take a unit $v \in O^\times_{Q,y}$ lifting $\pi / \prod_j t'^{e_j}_j \in O^\times_{Y,y}$ and put $u = g^* v \prod_{i,j} u_{ij}^{e_{ij} e_j} \in O^\times_{P,x}$. Then shrinking P and Q if necessary, we may assume U is a divisor of P defined by $\pi - u \prod_i T_i^{e_i}$ and V is a divisor of Q defined by $\pi - v \prod_j T'^{e_j}_j$. Since $g^*(\pi - v \prod_j T'^{e_j}_j) = \pi - u \prod_i T_i^{e_i}$, the assertion is proved.

We show the second diagram is cartesian. The question is local on X and Y. We take open subschemes $U' \subset X$ and $V' \subset Y$ and charts $M = \Gamma(X, \bar{M}_X) \to \Gamma(U', \bar{M}_X)$ and $M' = \Gamma(Y, \bar{M}_Y) \to \Gamma(V', \bar{M}_X)$ lifting the standard frames as in the proof of Proposition 3.15.2. Then by the proof there and in the notation above, the immersions $(U \times_S U')^\sim \to (P \times_S U')^\sim$ and $(V \times_S V')^\sim \to (Q \times_S V')^\sim$ are defined by the equations $1 - (u \prod_i T_i^{e_i} \otimes 1)/\pi$ and $1 - (v \prod_j T'^{e_j}_j \otimes 1)/\pi$ respectively. Since $u \prod_i T_i^{e_i} = g^*(v \prod_j T'^{e_j}_j)$, the second diagram is also cartesian.

Proof of Proposition 3.31. The assertion is local on $(X \times_S X)^\sim$. First we show the assertion on the generic fiber. Since the log structure is trivial on the generic fiber, it is sufficient to show that the map $f \times f : X \times X \to Y \times Y$ is of finite tor-dimension. Since X and Y are regular, the map $f : X \to Y$ is of finite tor-dimension. Since X and Y are flat over S, the map $f \times f$ is of finite tor-dimension.

We show the assertion on a neighborhood of the closed fiber. Let $x \in X$ be a point in the closed fiber and take an open neighboorhood U of x and V of $y = f(x)$ as in Lemma 3.32. It is enough to show that the map $(U \times_S X)^\sim \to (V \times_S Y)^\sim$ is of finite tor-dimension. In the second cartesian diagram in Lemma 3.32 (2) above, the right vertical arrow is of finite tor-dimension since $(P \times_S X)^\sim$ and $(Q \times_S Y)^\sim$ are regular. The horizontal arrows are regular immersions of codimension 1. Therefore, it is sufficient to apply the following Lemma.

Lemma 3.33 *Let*

$$\begin{array}{ccc} X & \xrightarrow{i} & P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[j]{} & Q \end{array}$$

be a cartesian diagram of noetherian schemes. Assume that i and j are regular immersions of the same codimension and g is of tor-dimension $\leq n$. Then f is also of tor-dimension $\leq n$.

Proof. The question is local. If $\mathcal{K}$ is a Koszul resolution of j_*O_Y by free O_Q-modules, the pull-back $g^*\mathcal{K}$ is a Koszul resolution of i_*O_X. Hence we have $i_*O_X = j_*O_Y \otimes^L_{O_Q} O_P$. Therefore, for a coherent O_Y-module $\mathcal{F}$, we have

$$i_*L^q f^*\mathcal{F} = \mathcal{H}_q(\mathcal{F} \otimes^L_{O_Y} O_X) = \mathcal{H}_q(\mathcal{F} \otimes^L_{O_Q} O_P) = L^q g^* j_*\mathcal{F}.$$

Now the assertion follows.

Corollary 3.34 *Let K be a discrete valuation field with perfect residue field, let X and Y be flat and regular schemes over $S = \text{Spec } O_K$ such that the generic fibers are smooth and that the reduced closed fibers have simple normal crossings. Let $f : X \to Y$ be a morphism over S. Then we have a commutative diagram*

$$\begin{array}{ccc} G(Y_K \times_K Y_K) & \xrightarrow{[[\ ,Y]]} & G(Y_s) \\ {\scriptstyle (f_K\times f_K)^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ G(X_K \times_K X_K) & \xrightarrow[{[[\ ,X]]}]{} & G(X_s). \end{array}$$

Proof. It is enough to show that the diagram

$$\begin{array}{ccc} G((Y \times_S Y)^\sim) & \xrightarrow{[[\ ,Y]]} & G(Y_s) \\ {\scriptstyle (f\times f)^{\sim *}}\downarrow & & \downarrow{\scriptstyle f^*} \\ G((X \times_S X)^\sim) & \xrightarrow[{[[\ ,X]]}]{} & G(X_s) \end{array}$$

is commutative since $G((Y \times_S Y)^\sim) \to G(Y_K \times_K Y_K)$ is surjective. The vertical arrows are defined since $(f \times f)^\sim$ and f are of finite tor-dimension. We show that both of the compositions are equal to $[[\ ,X]]_{(Y\times_S Y)^\sim}$ by applying Corollary 2.25. We apply Corollary 2.25.1 by taking Y to be S in Corollary 2.25.1, $(Y \times_S Y)^\sim$ to be X and $f : X \to Y$ to be $g : W' \to W$. Since f is of finite tor-dimension, the assumption of Corollary 2.25.1 is satisfied. Thus the composition $f^* \circ [[\ ,Y]]$ is equal to $[[\ ,X]]_{(Y\times_S Y)^\sim}$. We apply Corollary 2.25.3 by taking further X to be S' in Corollary 2.25.3, $(X \times_S X)^\sim$ to be $X' = W$ and the diagonal X to be V'. Since $(f \times f)^\sim$ and f is of finite tor-dimension, the assumption of Corollary 2.25.3 is satisfied. Thus the composition $[[X,\ ]] \circ (f \times f)^{\sim *}$ is equal to $[[\ ,X]]_{(Y\times_S Y)^\sim}$. Hence the diagram is commuative.

We deduce Lemma 1.1 from Corollary 3.34.

Proof of Lemma 1.1. We apply Corollary 3.34 to $f : X = \text{Spec } O_M \to Y = \text{Spec } O_L$. For $\sigma \in \text{Gal}(L/K)$ and $\tau \in \text{Gal}(M/K)$, let $Y_{\sigma,K} \subset Y_K \times_K Y_K$ be the graph of σ and let $X_{\tau,K} \subset X_K \times_K X_K$ be the graph of τ. We identify $G(X_s) = G(Y_s) = \mathbf{Z}$. By Corollary 2.22, we have $\text{sw}_{L/K}(\sigma) = [[Y_{\sigma,K}, Y]]$ and $\text{sw}_{M/K}(\tau) = [[X_{\tau,K}, X]]$. Since $(f_K \times f_K)^*([Y_{\sigma,K}]) = \sum_{\tau \mapsto \sigma}[X_{\tau,K}]$, the assertion follows from Corollary 3.34.

We prove that log products are flat.

Proposition 3.35 *Let L be a finite extension of K and assume the integer ring O_L is a discrete valuation ring and is finite over O_K. Let X and Y be regular and flat schemes of finite type over $T = \text{Spec } O_L$ such that the reduced closed fibers have simple normal crossings. Let $M = \mathbf{N}^m \to \Gamma(X, \bar{M}_X)$ and $M \to \Gamma(Y, \bar{M}_Y)$ be frames and $N = \mathbf{N} = \Gamma(T, \bar{M}_T) \to M$ be a map of frames with respect to both of X and Y. Let $(X \times_S Y)^\sim$ denote the log product with respect to the composite map of frames $N_S = \mathbf{N} = \Gamma(S, \bar{M}_S) \to N \to M$ and $(T \times_S T)^\sim$ be the log product with respect to the canonical map $N_S \to N$ of frames. Then the morphism $(X \times_S Y)^\sim \to (T \times_S T)^\sim$ is flat.*

Proof. The question is local on X and Y. It is clear on the generic fiber. We may assume that neither of the closed fibers X_s and Y_s is empty. Shrinking X if necessary, we may assume that there is a chart $M' = \Gamma(X, \bar{M}_X) \to \Gamma(X, M_X)$ lifting the standard frame of X and that the map $M \to M'$ is surjective. By Lemma 3.2.2, there is a non-empty subset $I \subset \{1, \ldots, m\}$ such that the map $M \to M'$ is identified with the projection $\mathbf{N}^m \to \mathbf{N}^I$. By renumbering, we may assume $I = \{1, \ldots, r\}$ and identify $M' = \mathbf{N}^r$ and $\mathbf{Z}[M'] = \mathbf{Z}[T_1, \ldots, T_r]$. By the proof of Lemma 1.17.2, further shrinking X if necessary, we may assume that X is a divisor defined by $\pi - u\prod_{i=1}^r T_i^{l_i}$ of a scheme P etale over $\mathbf{A}^n_{O_K} = \text{Spec } O_L[T_1, \ldots, T_n]$ for a unit $u \in \Gamma(P, O_P^\times)$. The scheme P is smooth over the monoid algebra $O_L[M'] = O_L[T_1, \ldots, T_r]$ and the composition map $M' \to O_L[M'] \to \Gamma(P, O_P) \to \Gamma(X, O_X)$ induces the chart $M' \to \Gamma(X, M_X)$. Shrinking X further if necessary, we define a faithfully flat scheme X_1 over X and a flat morphism $X_1 \to X_0 = \text{Spec} O_L \otimes_{\mathbf{Z}[N]} \mathbf{Z}[M']$ as follows.

Let $t_i \in \Gamma(X, O_X)$ be the image of $T_i \in \Gamma(P, O_P)$. We set $P_1 = P[u']/(u'^{l_1} - u)$ and $X_1 = X \times_P P_1$. The scheme P_1 is faithfully flat over P and hence X_1 is faithfully flat over X. We define a map $p : P_1 \to P_0 = \text{Spec } O_L[M']$ by $p^*(T_1) = u'T_1$ and $p^*(T_i) = T_i$ for $i > 1$. We show that the map $p : P_1 \to P_0$ is flat on a neighborhood of the divisor P_1' of P_1 defined by T_1. The divisor P_1' is flat over the divisor P' of P defined by T_1 and hence flat over the divisor P_0' of P_0 defined by T_1. The map $P_1 \to P_0$ is flat on a neighborhood of P_1' by [4] Chapitre 3 §5 Theorem 1 (3)⇒(1). Hence shrinking P if necessary, we may assume $P_1 \to P_0$ is flat. We naturally regard $X_0 = \text{Spec } O_L \otimes_{\mathbf{Z}[N]} \mathbf{Z}[M']$ as a subscheme of $P_0 = \text{Spec} O_L[M']$. Then we have $X_1 = X_0 \times_{P_0} P_1$ and the map $X_1 \to X_0$ is flat. We regard X_1 and X_0 as log schemes with the log structure induced by those on X and on P_0 respectively. The composite $M' \to O_L \otimes_{\mathbf{Z}[N]} \mathbf{Z}[M'] \to \Gamma(X_1, O_{X_1})$ induces the chart $M' \to \Gamma(Y, M_X) \to \Gamma(X_1, M_{X_1})$.

In the same way, shrinking Y if necessary, we may assume the following: There is a chart $M'' = \Gamma(Y, \bar{M}_Y) \to \Gamma(Y, M_Y)$ lifting the standard frame, there is a non-empty subset $J \subset \{1, \ldots, m\}$ such that the map $M \to M''$ is identified with the projection $\mathbf{N}^m \to \mathbf{N}^J$ and there is a faithfully flat scheme Y_1 over Y and a flat morphism $Y_1 \to Y_0 = \text{Spec } O_L \otimes_{\mathbf{Z}[N]} \mathbf{Z}[M'']$ such that the composite $M'' \to O_L \otimes_{\mathbf{Z}[N]} \mathbf{Z}[M''] \to \Gamma(Y_1, O_{Y_1})$ induces the chart $M'' \to \Gamma(Y, M_Y) \to \Gamma(Y_1, M_{Y_1})$.

We consider the log products $(X_1 \times_S Y_1)^\sim$ and $(X_0 \times_S Y_0)^\sim$ with respect to the maps $N_S \to M', N_S \to M''$ of frames. We have $(X_1 \times_S Y_1)^\sim = (X \times_S Y)^\sim \times_{X \times_S Y} (X_1 \times_S Y_1) = (X_0 \times_S Y_0)^\sim \times_{X_0 \times_S Y_0} (X_1 \times_S Y_1)$. Since $X_1 \to X$ and $Y_1 \to Y$ are faithfully flat, the map $(X_1 \times_S Y_1)^\sim \to (X \times_S Y)^\sim$ is faithfully flat. Since $X_0 \to X$ and $Y_0 \to Y$ are flat, the map $(X_1 \times_S Y_1)^\sim \to (X_0 \times_S Y_0)^\sim$ is flat. Therefore, it is reduced to show that the map $(X_0 \times_S Y_0)^\sim \to (T \times_S T)^\sim$ is flat.

By Corollary 3.10, we have

$$\begin{aligned}(X_0 \times_S Y_0)^\sim =&\mathrm{Spec}\left((O_L \otimes_{\mathbf{Z}[N]} \mathbf{Z}[M']) \otimes_{O_K} (O_L \otimes_{\mathbf{Z}[N]} \mathbf{Z}[M''])\right) \otimes_{\mathbf{Z}[M+M]} \mathbf{Z}[M + (M^{\mathrm{gp}}/N_S^{\mathrm{gp}})]\\ =&\mathrm{Spec}(O_L \otimes_{O_K} O_L) \otimes_{\mathbf{Z}[N+N]} \mathbf{Z}[M' + M''] \otimes_{\mathbf{Z}[M+M]} \mathbf{Z}[M + (M^{\mathrm{gp}}/N_S^{\mathrm{gp}})].\end{aligned}$$

Similarly, we have $(T \times_S T)^\sim = \mathrm{Spec}(O_L \otimes_{O_K} O_L) \otimes_{\mathbf{Z}[N+N]} \mathbf{Z}[N + (N^{\mathrm{gp}}/N_S^{\mathrm{gp}})]$. Thus it is sufficient to show that the monoid algebra homomorphism

$$\mathbf{Z}[N + N^{\mathrm{gp}}] \to \mathbf{Z}[M' + M''] \otimes_{\mathbf{Z}[M+M]} \mathbf{Z}[M + M^{\mathrm{gp}}] = \mathbf{Z}[(M' + M'') +_{M+M} (M + M^{\mathrm{gp}})]$$

is flat. In the right hand side, $(M' + M'') +_{M+M} (M + M^{\mathrm{gp}})$ denotes the amalgamate sum of monoids. To prove the flatness, we apply the following criterion.

Lemma 3.36 ([18] Proposition (4.1)) *For a morphism of integral monoids $h : N \to M$, the following condition (1) is equivalent to the combination of (2-i) and (2-ii).*

(1) $h_* : \mathbf{Z}[N] \to \mathbf{Z}[M]$ is flat.

(2-i) h is injective.

(2-ii) For $a_1, a_2 \in N, b_1, b_2 \in M$ satisfying $h(a_1)b_1 = h(a_2)b_2$, there exist $a_3, a_4 \in N, b \in M$ satisfying $b_1 = h(a_3)b, b_2 = h(a_4)b$ and $a_1a_3 = a_2a_4$.

We continue the proof of Proposition 3.35. To apply Lemma 3.36 to the map $h : N + N^{\mathrm{gp}} \to \tilde{M} = (M' + M'') +_{M+M} (M + M^{\mathrm{gp}})$, we verify that the monoid $\tilde{M}$ is integral. Decomposing it into a direct sum over $\{1, \ldots, r\}$, we may assume $M = \mathbf{N}$. Then M' and M'' are either 0 or M. The monoid $\tilde{M}$ is isomorphic to $M + M^{\mathrm{gp}}$ if $M' = M'' = M$, to M^{gp} if one is M and the other is 0, and is 0 if $M' = M'' = 0$. In any case $\tilde{M}$ is integral.

We verify that the conditions (2-i) and (2-ii) are satisfied. Since $\tilde{M}^{\mathrm{gp}} = M'^{\mathrm{gp}} \oplus M''^{\mathrm{gp}}$ and neither of $M'^{\mathrm{gp}} = \mathbf{Z}^I$ and $M''^{\mathrm{gp}} = \mathbf{Z}^J$ is 0, the map $h : N + N^{\mathrm{gp}} \to \tilde{M}$ is injective. Using $N = \mathbf{N}$, it is easy to verify the condition (2-ii) is satisfied. Thus, by applying Lemma (2) $\Rightarrow$ (1), we see that the map $\mathbf{Z}[N + N^{\mathrm{gp}}] \to \mathbf{Z}[\tilde{M}]$ is flat.

We study log products of semi-stable schemes. Let K be a discrete valuation field with perfect residue field. A scheme X locally of finite type over the integer ring O_K is said to be strictly semi-stable (resp. semi-stable), if the following conditions 1-3 are satisfied.

1. X is regular and flat over S.
2. The generic fiber X_K is smooth.
3. The closed fiber is a divisor with simple normal crossings (resp. normal crossings).

The condition 3 means that the closed fiber is reduced and that the condition (S) in Theorem 1.15 (resp. (N) in Theorem 1.10) is satisfied. A scheme X is strictly semi-stable (resp. semi-stable) over S, if and only if Zariski locally (resp. etale locally) it is etale over Spec $O_K[T_1, \ldots, T_n]/(T_1 \cdot \cdots \cdot T_n - \pi)$ where π is a prime element of K. The standard log structure on a strictly semi-stable scheme X over S is that defined by the closed fiber.

Lemma 3.37 *1. For a log smooth scheme X over S, the following conditions are equivalent.*

(1) X is strictly semi-stable and has the standard log structure.

(2) There exist a frame $M \to \Gamma(X, \bar{M}_X)$, a map $\mathbf{N} = \Gamma(S, \bar{M}_S) \to M$ of frames and an isomorphism $M/M^\times \to \mathbf{N}^r$ of monoids such that the composition $\mathbf{N} \to M \to \mathbf{N}^r$ sends 1 to $(1, \ldots, 1)$.

2. Let X and Y be strictly semi-stable schemes, $M \to \Gamma(X, \bar{M}_X), M \to \Gamma(Y, \bar{M}_Y)$ be frames and let $\mathbf{N} = \Gamma(S, \bar{M}_S) \to M$ be a map of frames with respect to both of X and Y. Then the log product $(X \times_S Y)^\sim$ is strictly semi-stable. The projections $(X \times_S Y)^\sim \to X, (X \times_S Y)^\sim \to Y$ are smooth. When $X = Y$, the log diagonal map $X \to (X \times_S X)^\sim$ is a regular immersion.

Proof. 1. To show (1) ⇒ (2), it is sufficient to take the standard frame. We show (2) ⇒ (1). By Corollary 3.7, we may assume that $M^\times = 1$ and hence $M = \mathbf{N}^r$. By Proposition 3.5, shrinking X if necessary, we may assume the frame $M \to \Gamma(X, \bar{M}_X)$ is induced by a chart $M \to \Gamma(X, M_X)$. Then by Lemma 3.4, X is regular and the log structure M_X is defined by a divisor D with simple normal crossings. By the assumption that 1 is sent to $(1, \ldots, 1)$, the divisor D is equal to the closed fiber. Since X is log smooth and the log structure is trivial on the generic fiber, the generic fiber is smooth.

2. The projections are strict and log smooth. Hence they are classically smooth. Since $(X \times_S Y)^\sim$ is smooth over a strictly semi-stable scheme, it is also strictly semi-stable. The log diagonal map is a section of a smooth map and is a regular immersion.

For X and Y as in Lemma 3.37.2, we define a log blow-up of the fiber product $X \times_S Y$ containing $(X \times_S Y)^\sim$ as an open subscheme as follows. Let $M = \mathbf{N}^r +_{\mathbf{N}} \mathbf{N}^r$ be the amalgamate sum of $\mathbf{N} \to \mathbf{N}^r : 1 \mapsto (1, \ldots, 1)$. We define a cone decomposition Σ of the dual monoid $N = Hom_{\text{monoid}}(M, \mathbf{N})$ of the monoid $M = \mathbf{N}^r +_{\mathbf{N}} \mathbf{N}^r$. Let $e_i, f_j, (i, j = 1, \ldots, r)$ be the standard generators of M. We put $I_r = \{1, \ldots, r\}$ and, for $(i, j) \in I_r \times I_r$, let $e^*_{i,j} \in N$ be the element characterized by $e^*_{i,j}(e_{i'}) = \delta_{ii'}$ and $e^*_{i,j}(f_{j'}) = \delta_{jj'}$. The monoid N is generated by $e^*_{i,j}$ for $(i, j) \in I_r \times I_r$. We consider the set $I_r \times I_r$ as a partially ordered set with the product partial order. Namely, we have $(i, j) \leq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$. We say a subset $\sigma \subset I_r \times I_r$ is a face of $I_r \times I_r$ if any two element of σ is comparable with respect to this partial order and let Σ be the set of faces.

Lemma 3.38 *Let $r \geq 1$ be an integer and Σ be as above.*

1. Under the notation above, the set of faces Σ of $I_r \times I_r$ defines a regular cone decomposition of N.

2. Let X and Y be strictly semi-stable schemes over S and let $\mathbf{N}^r \to \Gamma(X, \bar{M}_X), \mathbf{N}^r \to \Gamma(Y, \bar{M}_Y)$ be frames. Assume that the map $\varphi : \mathbf{N} = \Gamma(S, \bar{M}_S) \to \mathbf{N}^r$ sending 1 to $(1, \ldots, 1)$ is a map of frames with respect to both of X and Y. Let $(X \times_S Y)_\Sigma$ be the log blow-up of $X \times_S Y$ defined as in Lemma 3.11 and let $(X \times_S Y)^\sim$ be the log product with respect to the map of frames $\varphi : N \to \mathbf{N}^r$. Then the log blow-up $(X \times_S Y)_\Sigma$ is proper over $X \times_S Y$ and strictly semi-stable over S. It contains the log product $(X \times_S Y)^\sim$ as an open subscheme.

Proof. 1. We identify N with $\{((a_i), (b_j)) \in \mathbf{N}^{2r} | \sum_{i=1}^r a_i = \sum_{j=1}^r b_j\}$. For an element $k \in I_r \times \{1, 2\}$ and $((a_i), (b_j)) \in N$, we put $c_k = \sum_{i' \leq i} a_{i'}$ if $k = (i, 1)$ and $c_k = \sum_{j' \leq j} b_{j'}$ if $k = (j, 2)$ For a face σ, we define a surjection $g : I_r \times \{1, 2\} \to \{1, 2, \ldots, \#\sigma\}$ by $g(k) = \#\{(i', j) \in \sigma | i' \leq i\}$

if $k = (i, 1)$ and $= \{(i, j') \in \sigma | j' \leq j\}$ if $k = (j, 2)$. Then we have $N_\sigma = \{((a_i), (b_j)) \in N |\ c_k \geq c_{k'}$ if $g(k) \geq g(k')$ for $k, k' \in I_r \times \{1, 2\}\}$. Hence $N = \bigcup_{\sigma \in \Sigma} N_\sigma$. For $1 \leq l \leq \#\sigma$, we put $d_l = c_k$ for $k \in g^{-1}(l)$ and $d_0 = 0$. Then the map $N_\sigma \to \mathbf{N}^{\#\sigma} : ((a_i), (b_j)) \mapsto (d_l - d_{l-1})$ is an isomorphism. In fact,if $(i, j) \in \sigma$ is the l-th element with respect to the order, the image of $e^*_{i,j} \in N_\sigma$ is the l-th element of the standard basis of $\mathbf{N}^{\#\sigma}$. Hence Σ is regular.

2. By Lemma 3.11, the map $(X \times_S Y)_\Sigma \to X \times_S Y$ is proper and log etale. Hence $(X \times_S Y)_\Sigma$ is log smooth over S. We show that $(X \times_S Y)_\Sigma$ is strictly semi-stable by applying the criterion, Lemma 3.37.1. It is sufficient to show that $(X \times_S Y)_\sigma$ is strictly semi-stable for each face σ. Since N_σ is isomorphic to $\mathbf{N}^{\#\sigma}$, the monoid M_σ is isomorphic to $\mathbf{N}^{\#\sigma} \times \mathbf{Z}^{r-\#\sigma}$. Since $e^*_{i,j}(\varphi(1)) = 1$ for $(i, j) \in \sigma$, the image of $\varphi(1)$ by the composite $M \to M_\sigma / M_\sigma^\times \xrightarrow{\sim} \mathbf{N}^{\#\sigma}$ is $(1, \ldots, 1)$. Applying Lemma 3.37.1, we conclude $(X \times_S Y)_\sigma$ is strictly semi-stable. For the face $\sigma_0 = \{(i, i) | i \in I_r\}$, we have $M_{\sigma_0} = (\mathbf{N}^r +_{\mathbf{N}} \mathbf{N}^r)^\sim$ and $(X \times_S Y)_{\sigma_0} = (X \times_S Y)^\sim$.

3.6. Log Lefschetz trace formula.

We state and prove logarithmic Lefschetz trace formula. To state it, we introduce some notations. Let L be a discrete valuation field with perfect residue field E and let σ be an automorphism of the integer ring O_L of finite order. For a scheme W over $T = \text{Spec } O_L$, we define a T-scheme $W_\sigma \to T$ to be the base change $pr_2 : W \times_T T \to T$ by $\sigma : T \to T$. We assume that σ acts trivially on the residue field E and that the order of σ is a power of the characteristic p of E. We also assume that W is proper and semi-stable. For a prime number ℓ different from $p = \text{char } E$, we define a map $\sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell) \to H^*(W_{\sigma,\bar{L}}, \mathbf{Q}_\ell)$ as follows. Let L' be the henselization of L. The automorphism σ induces an automorphism of L' also denoted by σ. We put $L'_0 = \{a \in L' | \sigma(a) = a\}$. Then L' is a totally wildly ramified cyclic extension of L'_0. The Galois group G_{L'/L'_0} is equal to its wild inertia P_{L'/L'_0} and is generated by σ. We fix an embedding $L' \to \bar{L}$ and identify P_{L'/L'_0} with the quotient $P_{L'_0}/P_{L'}$ of the wild inertia subgroups. Let $\tilde{\sigma} \in P_{L_0}$ be a lifting of $\sigma \in P_{L'/L'_0} = P_{L'_0}/P_{L'}$. Then the map $1 \times \tilde{\sigma}^* : W_\sigma \times_L \bar{L} = W \times_L \bar{L} \to W \times_L \bar{L}$ induces a map $\tilde{\sigma}_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell) \to H^*(W_{\sigma,\bar{L}}, \mathbf{Q}_\ell)$. Since we assume W is proper and semi-stable and $\ell \neq p$, the action of the wild inertia $P_{L'}$ is trivial on $H^*(W_{\bar{L}}, \mathbf{Q}_\ell)$. Hence the map $\tilde{\sigma}_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell) \to H^*(W_{\sigma,\bar{L}}, \mathbf{Q}_\ell)$ is independent of the choice of a lifting $\tilde{\sigma}$. We define $\sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell) \to H^*(W_{\sigma,\bar{L}}, \mathbf{Q}_\ell)$ to be $\tilde{\sigma}_*$.

Assume W is strictly semi-stable of relative dimension d and put $M = \Gamma(W, \bar{M}_W)$ and $N = \Gamma(T, \bar{M}_T)$. Then, for an automorphism σ of O_L as above, the map $M = \Gamma(W, \bar{M}_W) \to \Gamma(W_\sigma, \bar{M}_{W_\sigma})$ defines a frame and the canonical map $N \to M$ is a map of frames with respect to both of W and W_σ. Hence the logarithmic fibered product $(W \times_T W_\sigma)^\sim$ is defined. Since $W_{\sigma,t} = W_t$ as a log scheme, the closed fiber $(W \times_T W_\sigma)^\sim_t = (W \times_T W_\sigma)^\sim \times_T t$ is canonically identified with $(W \times_T W)^\sim_t$. For an algebraic correspondence $\Gamma \in CH_d(W_L \times_L W_{\sigma,L})$, let Γ also denote its image in $Gr^F_d G(W_L \times_L W_{\sigma,L})$ by abuse of notation and let $\Gamma_t \in Gr^F_d G((W \times_T W)^\sim_t)$ denote the specialization $(\Gamma, t)_T$. Since the immersion $\Delta_{W_t} : W_t \to (W \times_T W)^\sim_t$ is a regular immersion by Lemma 3.37.2, the pull-back $\Delta^*_{W_t}(\Gamma_t) \in Gr^F_0 G(W_t)$ is defined. We define the degree map $\deg_{W_t} : G(W_t) \to G(t) = \mathbf{Z}$ to be the push-forward for $W_t \to t$.

Theorem 3.39 *Let L be a discrete valuation field with perfect residue field E of characteristic p and $\ell \neq p$ be a prime number. Let σ be an automorphism of O_L of order a power of p which induces the identity on the residue field E. Let W be a projective and strictly semi-stable scheme of relative* *dimension d over $T = \mathrm{Spec}\ O_L$. Then for an algebraic correspondence $\Gamma \in CH_d(W_L \times_L W_{\sigma,L})$, we have an equality of integers*

$$\mathrm{Tr}\ (\Gamma^* \circ \sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell)) = \deg_{W_t} \Delta^*_{W_t}(\Gamma_t).$$

Proof. We show the formula by using log-etale cohomology of the closed fiber. Basic references for log-etale cohomology are [7], [23], [24] and [15]. We regard t as a log scheme with the log structure induced by the standard one on T. The assumption on σ means that σ acts trivially on the log point t. Let $\bar{t}$ be a log geometric point over the log point t and $W_{\bar{t}}$ be the geometric closed fiber. Let $H^*_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)$ be the log-etale cohomology. By [24] Proposition (4.2), there is a canonical isomorphism $H^*(W_{\bar{L}}, \mathbf{Q}_\ell) \to H^*_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)$.

We fix an isomorphism $\mathbf{N}^r \to \Gamma(W, \bar{M}_W)$. It induces an isomorphism $\mathbf{N}^r \to \Gamma(W_\sigma, \bar{M}_{W_\sigma})$. Let $(W \times_T W_\sigma)^-$ be the log blow-up $(W \times_T W_\sigma)_\Sigma$ of $W \times_T W_\sigma$ studied in Lemma 3.38. It contains $(W \times_T W_\sigma)^\sim$ as an open subscheme. We reduce Theorem to a statement, Lemma 3.40 below, for an element in $Gr^d_F K((W \times_T W_\sigma)^-)$. Since W_L and $W_{\sigma,L}$ are projective and smooth, the chern character map $ch : Gr^d_F K(W_L \times_L W_{\sigma,L})_{\mathbf{Q}} \to CH_d(W_L \times_L W_{\sigma,L})_{\mathbf{Q}}$ is an isomorphism by Lemma 2.4.3. Since $(W \times_T W_\sigma)^-$ is regular by Lemma 3.37.2, the canonical map $K((W \times_T W_\sigma)^-) \to G((W \times_T W_\sigma)^-)$ is an isomorphism. Hence the maps $K((W \times_T W_\sigma)^-) \to K(W_L \times_L W_{\sigma,L})$ and $Gr^d_F K((W \times_T W_\sigma)^-) \to Gr^d_F K(W_L \times_L W_{\sigma,L})$ are surjective. Therefore, there exists an element $\tilde{\Gamma} \in Gr^d_F K((W \times_T W_\sigma)^-)_{\mathbf{Q}}$ such that the image of Γ in $CH_d(W_L \times_L W_{\sigma,L})_{\mathbf{Q}}$ is equal to $ch(\tilde{\Gamma}|_{W_L \times_L W_{\sigma,L}})$. Since the equality to be proved is an equality in $\mathbf{Q}_\ell$, we may assume that the image of Γ in $CH_d(W_L \times_L W_{\sigma,L})_{\mathbf{Q}}$ is the images of $\tilde{\Gamma} \in Gr^d_F K((W \times_T W_\sigma)^-)$ by replacing Γ by its multiple. The diagram

$$\begin{array}{ccccc}
Gr^d_F K(W_L \times_L W_{\sigma,L}) & \xrightarrow{ch} & CH_d(W_L \times_L W_{\sigma,L})_{\mathbf{Q}} & \longrightarrow & Gr^F_d G(W_L \times_L W_{\sigma,L})_{\mathbf{Q}} \\
\uparrow & & & & \downarrow{\scriptstyle (\ ,t)} \\
Gr^d_F K((W \times_T W_\sigma)^-) & \xrightarrow[(\ ,t)]{} & Gr^d_F K((W \times_T W_\sigma)^-_t) & \xrightarrow[\text{canores}]{} & Gr^F_d G((W \times_T W_\sigma)^\sim_t)_{\mathbf{Q}} \\
& & {\scriptstyle \Delta^*_{W_t}}\downarrow & & \downarrow{\scriptstyle \Delta^*_{W_t}} \\
& & Gr^d_F K(W_t) & \xrightarrow[\text{can}]{} & Gr^F_0 G(W_t)_{\mathbf{Q}}
\end{array}$$

is commutative, since the composition of the top horizontal arrows is the canonical map by Lemma 2.4.3. Hence the image of $\Delta^*_{W_t}(\Gamma_t) \in Gr^F_0 G(W_t)_{\mathbf{Q}}$ is the image of $\Delta^*_{W_t}(\tilde{\Gamma}_t) \in Gr^d_F K(W_t)$ where $\tilde{\Gamma}_t \in Gr^d_F K((W \times_T W_\sigma)^-_t)$ is the reduction of $\tilde{\Gamma}$. Thus Theorem is reduced to the following Lemma. Let $\deg : Gr^d_F K(W_t) \to \mathbf{Z}$ denote the composition map $Gr^d_F K(W_t) \to Gr^F_0 G(W_t) \stackrel{\deg}{\to} \mathbf{Z}$.

Lemma 3.40 *Let $\tilde{\Gamma}$ be an element of $Gr^d_F K((W \times_T W_\sigma)^-)$. Let $\Gamma \in CH_d(W_L \times_L W_{\sigma,L})_{\mathbf{Q}}$ be the chern character $ch(\tilde{\Gamma}|_{W_L \times_L W_{\sigma,L}})$ of the restriction and let $\Delta^*_{W_t}(\tilde{\Gamma}_t) \in Gr^d_F K(W_t)$ be the pull-back of the reduction $\tilde{\Gamma}_t \in Gr^d_F K((W \times_T W_\sigma)^-_t)$ of $\tilde{\Gamma}$. Then we have an equality of integers*

$$\mathrm{Tr}\ (\Gamma^* \circ \sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell)) = \deg_{W_t} \Delta^*_{W_t}(\tilde{\Gamma}_t).$$

We show that $\tilde{\Gamma}_t \in Gr^d_F K((W \times_T W_\sigma)^-_t)$ defines an endomorphism of $H^*_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)$ corresponding to $\Gamma^* \circ \sigma_*$ on $H^*(W_{\bar{L}}, \mathbf{Q}_\ell)$. We define an endomorphism $\tilde{\Gamma}^*_t$ of $H^*_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)$ as follows. The chern character map $ch : K((W \times_T W_\sigma)^-_t) \to H^{2d}_{\log}((W \times_T W_\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d))$ induces a map $ch : Gr^d_F K((W \times_T W_\sigma)^-_t) \to H^{2d}_{\log}((W \times_T W_\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d))$. It is the composition of the chern character map $ch : Gr^d_F K((W \times_T W_\sigma)^-_t) \to H^{2d}((W \times_T W_\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d))$ to the usual etale cohomology with the canonical map $H^{2d}((W \times_T W_\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d)) \to H^{2d}_{\log}((W \times_T W_\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d))$. We show that the projections $(W \times_T W_\sigma)^- \to W, (W \times_T W_\sigma)^- \to W_\sigma$ and the cup-product induce an isomorphism $\bigoplus_{p+q=r} H^p_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell(d)) \otimes H^q_{\log}(W_{\sigma,\bar{t}}, \mathbf{Q}_\ell) \to H^r_{\log}((W \times_T W_\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d))$. Since $(W \times_T W_\sigma)^-, W$ and W_σ are semi-stable, the log etale cohomology of the closed fibers are canonically isomorphic to the etale cohomology of the generic fibers by [24] Proposition (4.2). Since the canonical isomorphism is compatible with the pull-back and the cup-product, it is reduced to the Künneth formula for the generic fibers. We recall that the closed fiber $W_{\sigma,t}$ is identical with W_t as log schemes over t. By Poincaré duality loc.cit Theorem (7.5) for log-etale cohomology, we have a canonical isomorphism $\bigoplus_q \mathrm{End}(H^q_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)) \to H^{2d}_{\log}((W \times_T W_\sigma)^-_t, \mathbf{Q}_\ell(d))$. Taking the composition of the maps, we obtain a map $Gr^d_F K((W \times_T W_\sigma)^-_t) \to \bigoplus_q \mathrm{End}(H^q_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell))$. Thus an element $\tilde{\Gamma}_t \in Gr^d_F K((W \times_T W_\sigma)^-_t)$ defines an endomorphism $\tilde{\Gamma}^*_t$ of $H^q_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)$. It is the composition of

$$\begin{array}{ccccc} H^q_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell) = H^q_{\log}(W_{\sigma,\bar{t}}, \mathbf{Q}_\ell) & \xrightarrow{p_2^*} & H^q_{\log}((W \times_T W_\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell) & \xrightarrow{\cup ch(\tilde{\Gamma}_t)} & \\ H^{2d+q}_{\log}((W \times_T W_\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d)) & \xrightarrow{p_{1*}} & H^q_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell). & & \end{array}$$

We show that the endomorphism $\Gamma^* \circ \sigma_*$ of $H^*(W_{\bar{L}}, \mathbf{Q}_\ell)$ corresponds to the endomorphism Γ^*_t on $H^*_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)$.

Lemma 3.41 *Let the notation be as in Lemma 3.40. Let $\tilde{\Gamma}^*_t$ be the endomorphism of $H^q_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)$ defined above and let $ch(\Delta^*_{W_t}(\tilde{\Gamma}_t)) \in H^{2d}_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell(d))$ be the chern character of the pull-back $\Delta^*_{W_t}(\tilde{\Gamma}_t) \in Gr^d_F K(W_t)$. Then,*

1. *The diagram*

$$\begin{array}{ccc} H^*(W_{\bar{L}}, \mathbf{Q}_\ell) & \xrightarrow{\Gamma^* \circ \sigma_*} & H^*(W_{\bar{L}}, \mathbf{Q}_\ell) \\ {\scriptstyle \mathrm{can}}\downarrow & & \downarrow{\scriptstyle \mathrm{can}} \\ H^*_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell) & \xrightarrow[\Gamma^*_t]{} & H^*_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell) \end{array}$$

is commutative and we have an equality

$$\mathrm{Tr}\ (\Gamma^* \circ \sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell)) = \mathrm{Tr}\ (\tilde{\Gamma}^*_t : H^*_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)).$$

2. *We have an equality*

$$\mathrm{Tr}\ (\tilde{\Gamma}^*_t : H^*_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)) = \mathrm{Tr}\ (ch(\Delta^*_{W_t}(\tilde{\Gamma}_t))).$$

Proof. 1. It is sufficient to show the commutativity of the diagram

$$\begin{array}{ccccc} H^q(W_{\bar{L}}, \mathbf{Q}_\ell) & \xrightarrow{\sigma_*} & H^q(W_{\sigma,\bar{L}}, \mathbf{Q}_\ell) & \xrightarrow{p_2^*} & H^q(W_{\bar{L}} \times_{\bar{L}} W_{\sigma,\bar{L}}, \mathbf{Q}_\ell) \\ \downarrow & & \downarrow & & \downarrow \\ H^q_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell) & = & H^q_{\log}(W_{\sigma,\bar{t}}, \mathbf{Q}_\ell) & \xrightarrow{p_2^*} & H^q_{\log}((W \times_T W_\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell) \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{\cup[\Gamma]} & H^{2d+q}(W_{\bar{L}} \times_{\bar{L}} W_{\sigma,\bar{L}}, \mathbf{Q}_\ell(d)) & \xrightarrow{p_{1*}} & H^q(W_{\bar{L}}, \mathbf{Q}_\ell) \\ & \downarrow & & \downarrow \\ \xrightarrow{\cup ch(\tilde{\Gamma}_t)} & H^{2d+q}_{\log}((W \times_T W_\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d)) & \xrightarrow{p_{1*}} & H^q_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell). \end{array}$$

The vertical maps are the canonical isomorphisms. The commutativity of the first two squares is the functoriality of the canonical isomorphisms. The commutativity of the last square follows from the functoriality and the compatibility with the Poincaré duality. We show the remaining square is also commutative. The diagram

$$\begin{array}{ccccc} Gr^d_F K(W_L \times_L W_{\sigma,L}) & \xrightarrow{ch} & CH^d(W_L \times_L W_{\sigma,L})_{\mathbf{Q}} & \xrightarrow{cl} & H^{2d}(W_{\bar{L}} \times_{\bar{L}} W_{\sigma,\bar{L}}, \mathbf{Q}_\ell(d)) \\ \uparrow & & & & \downarrow \\ Gr^d_F K((W_T \times_T W_\sigma)^-) & \longrightarrow & Gr^d_F K((W \times_T W_\sigma)^-_t) & \xrightarrow[ch]{} & H^{2d}_{\log}((W \times_T W_\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d)) \end{array}$$

is commuatitive, since the composition of the upper horizontal arrow is the chern character map. Hence it follows from the compatiblity of the canonical isomorphism with the cup-product.

The equality is an immediate consequence of the commutative diagram.

2. By the functoriality of the chern character map, Künneth formula and Poincaré duality, we have a commutative diagram

$$\begin{array}{ccccc} Gr^d_F K((W \times_T W_\sigma)^-_t) & \xrightarrow{ch} & H^{2d}_{\log}((W \times_T W_\sigma)^-_{\bar{t}}, \mathbf{Q}_\ell(d)) & \longrightarrow & \bigoplus_q \mathrm{End}(H^q_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell)) \\ \Delta^* \downarrow & & \Delta^* \downarrow & & \downarrow \sum_q (-1)^q \mathrm{Tr} \\ Gr^d_F K(w_t) & \xrightarrow[ch]{} & H^{2d}_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell(d)). & \xrightarrow[\mathrm{Tr}]{} & \mathbf{Q}_\ell. \end{array}$$

The equality follows from this immediately.

To complete the proof of Theorem, we compare the trace map with the degree map.

Lemma 3.42 *Let* Γ *be an element in* $Gr^d_F K(W_t)$ *and let* $ch(\Gamma)$ *be the image by the chern character map* $ch : Gr^d_F K(W_t) \to H^{2d}_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell(d))$. *Then we have* $\mathrm{Tr}\ (ch(\Gamma)) = \deg \Gamma$. *In other words, we have a commutative diagram*

$$\begin{array}{ccc} Gr^d_F K(W_t) & \xrightarrow{ch} & H^{2d}_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell(d)) \\ \deg \downarrow & & \downarrow \mathrm{Tr} \\ \mathbf{Z} & \longrightarrow & \mathbf{Q}_\ell. \end{array}$$

Proof. Let $\pi : \bar{W}_t \to W_t$ be the normalization of W_t. It is projective and smooth over t. We show that the diagram

$$\begin{array}{ccccc} Gr^d_F K(W_t) & \xrightarrow{ch} & H^{2d}(W_{\bar{t}}, \mathbf{Q}_\ell) & \longrightarrow & H^{2d}_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell) \\ {\scriptstyle \pi^*}\downarrow & & {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}} \\ Gr^d_F K(\bar{W}_t) & \xrightarrow[ch]{} & H^{2d}(\bar{W}_{\bar{t}}, \mathbf{Q}_\ell) & \xrightarrow[\mathrm{Tr}]{} & \mathbf{Q}_\ell \end{array}$$

is commutative. Let $W^\circ_{\bar{t}}$ be the smooth locus of $W_{\bar{t}}$. Then the canonical map $H^{2d}_c(W^\circ_{\bar{t}}, \mathbf{Q}_\ell) \to H^{2d}(W_{\bar{t}}, \mathbf{Q}_\ell)$ is an isomorphism. The composition $H^{2d}_c(W^\circ_{\bar{t}}, \mathbf{Q}_\ell) \to H^{2d}(\bar{W}_{\bar{t}}, \mathbf{Q}_\ell) \to \mathbf{Q}_\ell$ is the trace map for $W^\circ_{\bar{t}}$. The other composition $H^{2d}_c(W^\circ_{\bar{t}}, \mathbf{Q}_\ell) \to H^{2d}_{\log}(W_{\bar{t}}, \mathbf{Q}_\ell) \to \mathbf{Q}_\ell$ is also equal to the trace map for $W^\circ_{\bar{t}}$ by the definition of the trace map for log etale cohomology in [23] Proof of Proposition (7.8.2). Hence the right square is commutative. The left square is commutative by the functoriality of the chern character map.

We show the equality Tr $(ch(\Gamma)) =$ deg Γ. Since the composition of the upper line of the commutative diagram above is the chern character map, we have Tr $(ch(\Gamma)) =$ Tr $(\pi^*(ch(\Gamma)))$. On the other hand, we have $\Gamma = \pi_*\pi^*\Gamma \in Gr^F_0 G(W_t)$ since $\pi_*[O_{\bar{W}_t}] = [O_{W_t}]$ mod $F_{d-1}G(W_t)$. Hence we have $\deg_{W_t}\Gamma = \deg_{W_t}\pi_*\pi^*\Gamma = \deg_{\bar{W}_t}\pi^*\Gamma$. Thus it is reduced to the well-known equality Tr $(ch(\pi^*\Gamma)) = \deg_{\bar{W}_t}\pi^*\Gamma$ for a projective smooth scheme $\bar{W}_t$.

We complete the proof of Theorem. By Lemma 3.41, we have Tr $(\Gamma^* \circ \sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell)) =$ Tr $ch(\Delta^*_{W_t}(\tilde{\Gamma}_t))$. Further, by Lemma 3.42 applied to $\Delta^*_{W_t}(\tilde{\Gamma}_t) \in Gr^d_F K(W_t)$, we have an equality Tr $ch(\Delta^*_{W_t}(\tilde{\Gamma}_t)) = \deg \Delta^*_{W_t}(\tilde{\Gamma}_t)$.

4. Proof of Conductor formula.

In this section, we prove Theorem 3.25 and hence complete the proof of Theorems 1.10 and 1.15, using alteration by de Jong. By Corollary 3.29, we may assume K is complete. In this section, K will denote a complete discrete valuation field with perfect residue field.

4.1. Rationality.

To prove Theorem 3.25, we apply the following weaker version of [6] Theorem 5.9.

Theorem 4.1 *Let K be a complete discrete valuation field with perfect residue field and X be a proper and flat scheme over the integer ring $S = \mathrm{Spec} O_K$. Then there exist a finite normal extension L of K, a strictly semi-stable and projective scheme W over the integer ring $T = \mathrm{Spec}\ O_L$ and a proper, surjective and generically finite morphism $f : W \to X$ over S.*

By [22], we have the following corollary.

Corollary 4.2 *Let X_K be a proper scheme over K. Then there exist a finite normal extension L of K, a strictly semi-stable and projective scheme W over $T = \mathrm{Spec}\ O_L$ and a proper, surjective and generically finite morphism $f_K : W_L \to X_K$ over K.*

To show Theorem 3.25, we define a map substituting for the map $[[\ , X]] : G(X_K \times_K X_K) \to G(X_s)$, using an alteration. Let K be a complete discrete valuation field with perfect residue field and let X_K be a smooth scheme over K. Let L be a totally ramified finite normal extension, W be a strictly semi-stable scheme over the integer ring $T = \mathrm{Spec}\ O_L$ and let $f_K : W_L \to X_K$ be a morphism over K. Let t be the closed point of T. We define a map

$$[[\ , W]]' : G(X_K \times_K X_K) \to G(W_t).$$

When we have a regular and flat scheme X over O_K with an isomorphism $X \otimes_{O_K} K \to X_K$ such that the reduced closed fiber is a divisor with normal crossings and the map $W_L \to X_K$ is extended to a map $W \to X$, it is related to the map $[[\ , X]] : G(X_K \times_K X_K) \to G(X_s)$ as to be shown in Proposition 4.5 in the next subsection.

The definition is as follows. Let L_0 be the separable closure of K in L. It is a finite Galois extension of K. Let P_0 be the wild inertia subgroup of the Galois group $G_0 = \mathrm{Gal}(L_0/K)$. We naturally identify G_0 with $\mathrm{Aut}_K(L)$. For $\sigma \in G_0$, let $pr_2 : W_\sigma = W \times_T T \to T$ be the base change of $W \to T$ by $\sigma : T \to T$ and $f_\sigma : W_\sigma \to X$ be the map $f \circ (1 \times \sigma)$. The specialization map $(\ ,t)_T : G(W_L \times_L W_{\sigma,L}) \to G((W \times_T W)^\sim_t)$ for $\sigma \in P_0$ and the pull-back map $\Delta^*_{W_t} : G((W \times_T W)^\sim_t) \to G(W_t)$ are defined. Let $q = [L : L_0]$ be the inseparable degree and define a map $[[\ , W]]' : G(X_K \times_K X_K) \to G(W_t)$ to be the composition map

$$G(X_K \times_K X_K) \xrightarrow{((f_K \times f_{\sigma,K})^*)_\sigma} \bigoplus_{\sigma \in P_0} G(W_L \times_L W_{\sigma,L}) \xrightarrow{-q\sum_{\sigma\in P_0} \mathrm{sw}(\sigma)(\ ,t)_T} G((W \times_T W)^\sim_t) \xrightarrow{\Delta^*_{W_t}} G(W_t).$$

Namely, we put $[[\Gamma, W]]' = -q \sum_{\sigma \in P_0} \mathrm{sw}(\sigma) \cdot \Delta^*_{W_t}(\Gamma_{\sigma,t})$ for $\Gamma \in G(X_K \times_K X_K)$. The degree map $\deg_{W_t} : G(W_t) \to G(t) = \mathbf{Z}$ is the push-forward for $W_t \to t$.

Theorem 3.25.1 is reduced to the following Lemma.

Lemma 4.3 *Let K be a complete discrete valuation field with perfect residue field and X_K be a smooth scheme of dimension d over K. Let L be a totally ramified finite normal extension of K, W be a strictly semi-stable and quasi-projective scheme over the integer ring $T = \text{Spec } O_L$ and let $f_K : W_L \to X_K$ be a proper, surjective and generically finite morphism over K. Then,*

1. The map $[[\ ,W]]' : G(X_K \times_K X_K) \to G(W_t)$ preserves the topological filtrations.

2. Assume X_K is proper over K and W is projective over T as in Corollary 4.2. For $\Gamma \in CH_d(X_K \times_K X_K)$, let $[[\Gamma, W]]'$ also denote the image of Γ by the composition $CH_d(X_K \times_K X_K) \to Gr^F_d G(X_K \times_K X_K) \overset{[[\ ,W]]'}{\to} Gr^F_0 G(W_t)$. Then we have

$$[W : X]\text{Sw}(\Gamma, X_K/K) = -\deg_{W_t}[[\Gamma, W]]'.$$

Proof of Lemma 4.3 ⇒ Theorem 3.25.1. By Corollary 3.30, we may assume K is complete. Let L, W and $W_L \to X_K$ be as in Corollary 4.2. By Corollary 3.30, we may replace K by its maximum unramified extension in L. Hence we may assume L is a totally ramified normal extension. Thus the assumption of Lemma 4.3.2 is satisfied. Hence the assertion follows.

Proof of Lemma 4.3. 1. It follows from the definition of the map $[[\ ,W]]'$ and Lemmas 2.2.1 and 2.7.

2. We deduce it from the following consequence of the log Lefschetz trace formula, Theorem 3.39.

Lemma 4.4 *Let K be a complete discrete valuation field with perfect residue field F and let X_K be a proper smooth scheme of dimension d over K. Let σ be an element in the wild inertia P_K and $\Gamma \in CH_d(X_K \times_K X_K)$ be an algebraic correspondence. Let $\ell \neq p$ be a prime number. Then,*

1. The trace $\text{Tr}(\Gamma^* \circ \sigma : H^*(X_{\bar{K}}, \mathbf{Q}_\ell))$ is independent of ℓ.

Let L be a totally ramified finite normal extension of K, W be a strictly semi-stable and projective scheme over the integer ring $T = \text{Spec } O_L$ and let $f_K : W_L \to X_K$ be a proper, surjective and generically finite morphism over K as in Corollary 4.2. Then,

2. The image of the wild inertia P_L in G_K acts trivially on $H^*(X_{\bar{K}}, \mathbf{Q}_\ell)$.

3. Let σ also denote the induced automorphism of L and let $f_\sigma : W_{\sigma,L} \to X_K$ denote the composition $f \circ (1 \times \sigma)$. We put $\Gamma_\sigma = (f \times f_\sigma)^*\Gamma \in CH_d(W_L \times_L W_{\sigma,L})$. Then, we have an equality

$$[W_L : X_L]\text{Tr } (\Gamma^* \circ \sigma_* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell)) = \deg_{W_t} \Delta^*_{W_t}(\Gamma_{\sigma,t}).$$

Proof. 2. Since W is proper and semi-stable over T, the wild inertia P_L acts trivially on $H^*(W_{\bar{L}}, \mathbf{Q}_\ell)$. The composition $f_* \circ f^* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell) \to H^*(W_{\bar{L}}, \mathbf{Q}_\ell) \to H^*(X_{\bar{K}}, \mathbf{Q}_\ell)$ is the multiplication by $[W_{\bar{L}} : X_{\bar{K}}] = [W_L : X_L]$ on $H^*(X_{\bar{K}}, \mathbf{Q}_\ell)$. Hence the G_L-equivariant map $f^* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell) \to H^*(W_{\bar{L}}, \mathbf{Q}_\ell)$ is injective and the assertion follows.

3. By Theorem 3.39, it is enough to show the equality

$$[W_L : X_L]\text{Tr } (\Gamma^* \circ \sigma_* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell)) = \text{Tr } (\Gamma^*_\sigma \circ \sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell)).$$

We prove the equality by showing the commutativity of the diagram.

$$\begin{array}{ccccc} H^*(W_{\bar{L}}, \mathbf{Q}_\ell) & \xrightarrow{\sigma_*} & H^*(W_{\sigma,\bar{L}}, \mathbf{Q}_\ell) & \xrightarrow{\Gamma^*_\sigma} & H^*(W_{\bar{L}}, \mathbf{Q}_\ell) \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle f_{\sigma*}}\downarrow & & {\scriptstyle f^*}\uparrow \\ H^*(X_{\bar{K}}, \mathbf{Q}_\ell) & \xrightarrow[\sigma_*]{} & H^*(X_{\bar{K}}, \mathbf{Q}_\ell) & \xrightarrow[\Gamma^*]{} & H^*(X_{\bar{K}}, \mathbf{Q}_\ell). \end{array}$$

We show the commutativity. By the functoriality of the cycle map, the pull-back of the class $[\Gamma] \in H^{2d}(X_{\bar{K}} \times X_{\bar{K}}, \mathbf{Q}_\ell(d))$ by $f \times f_\sigma$ is $[\Gamma_\sigma] \in H^{2d}(W_{\bar{L}} \times W_{\sigma,\bar{L}}, \mathbf{Q}_\ell(d))$. Therefore, by the definition of the map $\Gamma^* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell) \to H^*(X_{\bar{K}}, \mathbf{Q}_\ell)$ and of $\Gamma^*_\sigma : H^*(W_{\bar{L}}, \mathbf{Q}_\ell) \to H^*(W_{\sigma,\bar{L}}, \mathbf{Q}_\ell)$, the right square is commutative. The left square is commutative by the transport of structure.

We show the equality. Recall that the map $f^* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell) \to H^*(W_{\bar{L}}, \mathbf{Q}_\ell)$ is injective. By the commutative diagram above, the image $\mathrm{Im}\ \Gamma^*_\sigma \circ \sigma_*$ is a subspace of $\mathrm{Im}\ f^*$. Hence the trace $\mathrm{Tr}\ (\Gamma^*_\sigma \circ \sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell))$ is equal to the trace of the restriction on $\mathrm{Im}\ f^*$. Thus we obtain

$$\begin{aligned}\mathrm{Tr}\ (\Gamma^*_\sigma \circ \sigma_* : H^*(W_{\bar{L}}, \mathbf{Q}_\ell)) &= \mathrm{Tr}\ (\Gamma^* \circ \sigma_* \circ f_* \circ f^* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell)) \\ &= [W_L : X_L]\mathrm{Tr}\ (\Gamma^* \circ \sigma_* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell)).\end{aligned}$$

1. It follows from 3 and Corollary 4.2.

We return to the proof of Lemma 4.3.2. By the definition of the Swan conductor and by $[W : X] = q \cdot [W_L : X_L][L_0 : K]$, we have

$$[W : X]\mathrm{Sw}(\Gamma, X_K/K) = q \cdot [W_L : X_L] \sum_{\sigma \in P_0} \mathrm{sw}(\sigma)\mathrm{Tr}(\Gamma^* \circ \sigma_* : H^*(X_{\bar{K}}, \mathbf{Q}_\ell)).$$

On the other hand, for $\sigma \in P_0$, we put $\Gamma_\sigma = (f_K \times f_{\sigma,K})^*\Gamma \in G(W_L \times_L W_{\sigma,L})$ and $\Gamma_{\sigma,t} = (\Gamma_\sigma, t)_T \in G((W \times_T W)^\sim_t)$. Then by the definition of the map $[[\ , W]]'$, we have

$$-\deg_{W_t}[[\Gamma, W]]' = q \sum_{\sigma \in P_0} \mathrm{sw}(\sigma) \cdot \deg_{W_t} \Delta^*_{W_t}(\Gamma_{\sigma,t}).$$

Hence it follows from Lemma 4.4.3.

4.2. Equality.

We reduce Theorem 3.25.2 to the following Proposition.

Proposition 4.5 *Let K be a complete discrete valuation field with perfect residue field and X be a regular separated and flat scheme of finite type over $S = \mathrm{Spec}\ O_K$. Assume the generic fiber is smooth and the reduced closed fiber has simple normal crossings. Let L be a totally ramified finite normal extension of K and q be the inseparable degree of L over K. Let W be a quasi-projective and strictly semi-stable scheme over $T = \mathrm{Spec}\ O_L$ and let $f : W \to X$ be a morphism over S. Let f^* denote the map $(\ , W)_X : G(X_s) \to G(W_t)$. Then the map $q \cdot [[\ , W]]' : G(X_K \times_K X_K) \to G(W_t)$ is equal to the composition*

$$G(X_K \times_K X_K) \xrightarrow{q \cdot [[\ ,X]]} G(X_s) \xrightarrow{f^*} G(W_t).$$

Their compositions with the surjection $G((X \times_S X)^\sim) \to G(X_K \times_K X_K)$ are the same as the map

$$q \cdot [[\ , W]]_{(X \times_S X)^\sim} : G((X \times_S X)^\sim) \longrightarrow G(W_t).$$

Proof of Proposition 4.5 ⇒ Theorem 3.25.2. Let $X \to S$ be as in Theorem 3.25.2. By Corollary 3.30, we may assume K is complete. Let $T = \mathrm{Spec}\ O_L$ and $f : W \to X$ be as in Theorem 4.1.

We may replace K by its maximum unramified extention in L by Corollary 3.30. Hence we may assume L is totally ramified over K. Thus the assumption of Proposition 4.5 is satisfied.

Let $\Gamma \in CH_d(X_K \times_K X_K)$ be an algebraic correspondence. By Lemma 4.3.2, Proposition 4.5 and by Corollary 2.3.2, we have

$$\begin{aligned}[W:X]\mathrm{Sw}(\Gamma^*, X_K/K) &= -\deg_{W_s}[[\Gamma, W]]' = -\deg_{W_s} f^*[[\Gamma, X]] \\ &= -[W:X]\deg_{X_s}[[\Gamma, X]].\end{aligned}$$

Remark. If L is assumed separable over K, there is an alternative proof for Theorem 1.15 using the projection formula Proposition 2.19. It goes as:

$$\begin{aligned}[W:X]\mathrm{Sw}(X_K/K) &= -\deg_{W_s}[[X, W]]' = -\deg_{W_s}[[W, (f\times f)^{\sim *}X]]_{(W\times_S W)^\sim} \\ &= -\deg_{X_s}[[X, f_*W]] = -[W:X]\deg_{X_s}[[X, X]].\end{aligned}$$

Here, the first equality is Lemma 4.3.2, the second is proved similarly as Proposition 4.5, the third is Proposition 2.19.2 and the last equality follows from Corollary 3.19.2.

In the rest of paper, we prove Proposition 4.5 to complete the proof of Theorem 3.25.2. First we relate the logarithmic localized intersection product with the Swan character.

Lemma 4.6 *Let L_0 be the separable closure of K in L and let t_0 be the closed point of $T_0 =$* Spec O_{L_0}*. Let P_0 be the wild inertia subgroup of the Galois group $G_0 = \mathrm{Gal}(L_0/K)$. Then, the map $[[\ ,T_0]]_{(T_0\times_S T_0)^\sim} : G((W\times_S W)^\sim) \to G((W\times_{T_0} W)^\sim_{t_0})$ is equal to the composite map*

$$G((W\times_S W)^\sim) \to \bigoplus_{\sigma\in P_0} G(W_L \times_{L_0} W_{\sigma,L}) \xrightarrow{-\sum_{\sigma\in P_0}\mathrm{sw}(\sigma)(\ ,t_0)_{T_0}} G((W\times_{T_0} W)^\sim_{t_0}).$$

Proof. The map $\coprod_{\sigma\in G_0} T_{0,\sigma} \to T_0\times_S T_0$ is surjective and $(W\times_{T_0} W_\sigma)^\sim = (W\times_S W)^\sim \times_{(T_0\times_S T_0)^\sim} T_{0,\sigma}$. Hence the map $\coprod_{\sigma\in G_0}(W\times_{T_0} W_\sigma)^\sim \to (W\times_S W)^\sim$ is surjective and consequently the sum of the push-forward map $\bigoplus_{\sigma\in G_0} G((W\times_{T_0} W_\sigma)^\sim) \to G((W\times_S W)^\sim)$ is surjective. The diagram

$$\begin{array}{ccc} \bigoplus_{\sigma\in G_0} G((W\times_{T_0} W_\sigma)^\sim) & \xrightarrow{\oplus\mathrm{res}} & \bigoplus_{\sigma\in G_0} G(W_L\times_{L_0} W_{\sigma,L}) \\ \downarrow & & \downarrow \\ G((W\times_S W)^\sim) & \longrightarrow & \bigoplus_{\sigma\in P_0} G(W_L\times_{L_0} W_{\sigma,L}) \end{array}$$

is commutative. Hence, it is sufficient to show that the map $[[\ ,T_0]]_{(T_0\times_S T_0)^\sim} : G((W\times_{T_0} W_\sigma)^\sim) \to G((W\times_{T_0} W)^\sim_{t_0})$ is equal to the map $-\mathrm{sw}(\sigma)(\ ,t_0)_{T_0}$ if $\sigma\in P_0$ and is the 0-map if $\sigma \in G_0 - P_0$. In Corollary 2.25.2, we take T_0 to be S, $(T_0\times_S T_0)^\sim$ to be X, $(W\times_{T_0} W_\sigma)^\sim \to T_{0,\sigma}$ to be $g : W' \to W$ and $\Delta^\sim : T_0 \to (T_0\times_S T_0)^\sim$ to be $V\to X$. Since $T_{0,\sigma} = T_0$ is regular, the assumption of Corollary 2.25.2 is satisfied. Since $[[T_{0,\sigma}, T_0]]_{(T_0\times_S T_0)^\sim} = -\mathrm{sw}(\sigma) \in G(t_0) = \mathbf{Z}$ for $\sigma\in P_0$ and $T_{0,\sigma}\cap T_0 = \emptyset$ for $\sigma\in G_0 - P_0$ by Corollary 2.22.2, the assertion follows.

Now we can complete the proof of Proposition 4.5 in the case where $L = L_0$ is separable over K, by showing the following Lemma.

Lemma 4.7 *1. The diagram*

$$\begin{array}{ccc} G((X\times_S X)^\sim) & \xrightarrow{[[\ ,X]]} & G(X_s) \\ {\scriptstyle [[\ ,(W\times_T W)^\sim]]_{(X\times_S X)^\sim}}\big\downarrow & & \big\downarrow {\scriptstyle f^*} \\ G((W\times_T W)^\sim_t) & \xrightarrow[\Delta^*_{W_t}]{} & G(W_t) \end{array}$$

is commutative and the both compositions are equal to the map

$$[[\ ,W]]_{(X\times_S X)^\sim} : G((X\times_S X)^\sim) \longrightarrow G(W_t)$$

.

2. The map $[[\ ,(W\times_{T_0} W)^\sim]]_{(X\times_S X)^\sim} : G((X\times_S X)^\sim) \to G((W\times_{T_0} W)^\sim_{t_0})$ *is equal to the composite map*

$$G((X\times_S X)^\sim) \xrightarrow{(f\times f)^*} G((W\times_S W)^\sim) \xrightarrow{[[\ ,T_0]]_{(T_0\times_S T_0)^\sim}} G((W\times_{T_0} W)^\sim_{t_0}).$$

Corollary 4.8 *The diagram*

$$\begin{array}{ccccc} G((X\times_S X)^\sim) & \xrightarrow{(f\times f)^{\sim *}} & G((W\times_S W)^\sim) & \xrightarrow{[[\ ,T_0]]_{(T_0\times_S T_0)^\sim}} & G((W\times_{T_0} W)^\sim_{t_0}) \\ \big\downarrow & & \big\downarrow & & \big\| \\ G(X_K\times_K X_K) & \xrightarrow[((f_K\times f_{\sigma,K})^*)_\sigma]{} & \bigoplus_{\sigma\in P_0} G(W_L\times_{L_0} W_{\sigma,L}) & \xrightarrow[-\sum_{\sigma\in P_0}\mathrm{sw}(\sigma)(\ ,t_0)_{T_0}]{} & G((W\times_{T_0} W)^\sim_{t_0}). \end{array}$$

is commutative. The map $[[\ ,(W\times_{T_0} W)^\sim]]_{(X\times_S X)^\sim} : G((X\times_S X)^\sim) \to G((W\times_{T_0} W)^\sim_{t_0})$ *is equal to the composition of the upper horizontal arrows.*

Proof of Corollary. The commutativity of the left square is clear. That of the right square is Lemma 4.6. The last assertion on the composition is Lemma 4.7.2.

Proof of Lemma 4.7. 1. Since $\Delta : W \to (W\times_T W)^\sim$ is a section of a smooth morphism by Lemma 3.37, the map $\Delta_{W_t} : W_t \to (W\times_T W)^\sim_t$ is a regular immersion and the map $\Delta^*_{W_t} : G((W\times_T W)^\sim_t) \to G(W_t)$ is defined. We prove the assertion by applying Corollary 2.25.1. We take X to be S in Corollary 2.25.1 and $(X\times_S X)^\sim$ to be X. First we show that the composition via $G(X_s)$ is equal to the map $[[\ ,W]]_{(X\times_S X)^\sim}$. For this, we further take $X \to (X\times_S X)^\sim$ to be $f : W \to X$ and $f : W \to X$ to be $g : W' \to W$. Since X is regular, the map $f : W \to X$ is of finite tor-dimension and the assumption of Corollary 2.25.1 is satisfied. Hence the assertion follows.

We consider the other composition. We take $(f\times f)^\sim : (W\times_T W)^\sim \to (X\times_S X)^\sim$ to be $f : W \to X$ in Corollary 2.25.1 and $W \to (W\times_T W)^\sim$ as $g : W' \to W$. Since the log diagonal map $W \to (W\times_T W)^\sim$ is a regular immersion by Lemma 3.37.2, it is of finite tor-dimension. Hence the assumption of Corollary 2.25.1 is also satisfied for $W \to (W\times_T W)^\sim \to (X\times_S X)^\sim$. We apply it to complete the proof.

2. We prove it by applying Corollary 2.25.3. We take X to be S in Corollary 2.25.3, $(f \times f)^\sim : (W \times_S W)^\sim \to (X \times_S X)^\sim$ to be $f : W \to X$, T_0 to be S', $(W \times_S W)^\sim \to (T_0 \times_S T_0)^\sim$ to be $g : W \to X'$ and the log diagonal $T_0 \to (T_0 \times_S T_0)^\sim$ to be $V' \to X'$. Then $W' = W \times_{X'} V'$ is $(W \times_{T_0} W)^\sim$. The map $(f \times f)^\sim$ is of finite tor-dimension by Proposition 3.31 and $(W \times_S W)^\sim \to (T \times_S T)^\sim$ is flat by Proposition 3.35. The closed subset $Z_{W'}$ is a subset of the closed fiber $Z_{W'} = (W \times_{T_0} W)^\sim_{t_0}$. We put $Z_1 = Z_{W'}$. Then the assumption of Corollary 2.25.3 is satisfied. To complete the proof, it suffices to apply it.

Proof of Proposition 4.5. Since the map $G(((X \times_S X)^\sim) \to G(X_K \times_K X_K)$ is surjective, it is sufficient to show the last assertion. The assertion for the composition $q \cdot f^* \circ [[\ , X]]$, it is shown in Lemma 4.7.1. We show the assertion for $q \cdot [[\ , W]]'$. By Lemma 4.7.1 and by the definition of the map $[[\ , W]]'$, it is sufficient to show that the map $[[\ , (W \times_T W)^\sim]]_{(X \times_S X)^\sim} : G((X \times_S X)^\sim) \to G((W \times_T W)^\sim_t)$ is equal to the composition

$$\begin{array}{l} G((X \times_S X)^\sim) \\ \quad\downarrow \\ G(X_K \times_K X_K) \xrightarrow{((f_K \times f_{\sigma,K})^*)_\sigma} \bigoplus_{\sigma \in P_0} G(W_L \times_L W_{\sigma,L}) \xrightarrow{-q\sum_{\sigma \in P_0} \mathrm{sw}(\sigma)(\ ,t)_T} G((W \times_T W)^\sim_t). \end{array}$$

Hence, in the case where $L = L_0$ is separable, Proposition 4.5 follows from Corollary 4.8 since $T = T_0$ and $q = 1$.

We show the general case. By Lemma 4.9.1 below, it is sufficient to show that the diagram

$$\begin{array}{ccccc} G((X \times_S X)^\sim) & \xrightarrow{q \cdot [[\ ,(W \times_T W)^\sim]]} & G((W \times_T W)^\sim_t) & \xrightarrow{\mathrm{can}} & G((W \times_{T_0} W)^\sim_{t_0}) \\ \downarrow & & & & \uparrow \mathrm{can} \\ G(X_K \times_K X_K) & \xrightarrow[q \cdot ((f_K \times f_{\sigma,K})^*)_\sigma]{} & \bigoplus_{\sigma \in P_0} G(W_L \times_L W_{\sigma,L}) & \xrightarrow[-q\sum_{\sigma \in P_0} \mathrm{sw}(\sigma)(\ ,t)_T]{} & G((W \times_T W)^\sim_t) \end{array}$$

is commutative. We show that the both composition maps are equal to $[[\ , (W \times_{T_0} W)^\sim]]$. For the upper line, it is Lemma 4.9.2 below. For the other, it follows from Corollaries 4.8 and 4.10 below.

Lemma 4.9 *1. The immersion $(W \times_T W)^\sim \to (W \times_{T_0} W)^\sim$ induces an isomorphism $G((W \times_T W)^\sim_t) \to G((W \times_{T_0} W)^\sim_{t_0})$.*

2. The map $[[\ , (W \times_{T_0} W)^\sim]]_{(X \times_S X)^\sim} : G((X \times_S X)^\sim) \to G((W \times_{T_0} W)^\sim_{t_0})$ is equal to the composition

$$G((X \times_S X)^\sim) \xrightarrow{q \cdot [[\ ,(W \times_T W)^\sim]]} G((W \times_T W)^\sim_t) \xrightarrow{\mathrm{can}} G((W \times_{T_0} W)^\sim_{t_0}).$$

3. For $\sigma \in G$, the map $(f_K \times f_{\sigma,K})^* : G(X_K \times_K X_K) \to G(W_L \times_{L_0} W_{\sigma,L})$ is equal to the composition

$$G(X_K \times_K X_K) \xrightarrow{q \cdot (f_K \times f_{\sigma,K})^*} G(W_L \times_L W_{\sigma,L}) \xrightarrow{\mathrm{can}} G(W_L \times_{L_0} W_{\sigma,L})$$

4. The diagram

$$\begin{array}{ccc} G(W_L \times_L W_{\sigma,L}) & \xrightarrow{q(\ ,t)_T} & G((W \times_T W)^{\sim}_t) \\ {\scriptstyle\text{can}}\Big\downarrow & & \Big\downarrow{\scriptstyle\text{can}} \\ G(W_L \times_{L_0} W_{\sigma,L}) & \xrightarrow[(\ ,t_0)_{T_0}]{} & G((W \times_{T_0} W)^{\sim}_{t_0}) \end{array}$$

is commutative.

Corollary 4.10 *The diagram*

$$\begin{array}{ccccc} G(X_K \times_K X_K) & \xrightarrow{q\cdot((f_K \times f_{\sigma,K})^*)_\sigma} & \bigoplus_{\sigma \in P_0} G(W_L \times_L W_{\sigma,L}) & \xrightarrow{-q\cdot\sum_{\sigma\in P_0} \mathrm{sw}(\sigma)(\ ,t)_T} & G((W \times_T W)^{\sim}_t) \\ \Big\| & & {\scriptstyle\text{can}}\Big\downarrow & & \Big\downarrow{\scriptstyle\text{can}} \\ G(X_K \times_K X_K) & \xrightarrow[((f_K \times f_{\sigma,K})^*)_\sigma]{} & \bigoplus_{\sigma \in P_0} G(W_L \times_{L_0} W_{\sigma,L}) & \xrightarrow[-\sum_{\sigma\in P_0} \mathrm{sw}(\sigma)(\ ,t_0)_{T_0}]{} & G((W \times_{T_0} W)^{\sim}_{t_0}) \end{array}$$

is commutative.

Proof of Corollary 4.10. The left square is commutative by Lemma 4.9.3 and the right is by Lemma 4.9.4.

Proof of Lemma 4.9. Let π_L be a prime element of L. Since L is a purely inseparable extension of L_0 of degree q, $\pi_0 = \pi_L^q$ is a prime element of L_0. Since O_L is finite over O_K, the map $O_{L_0}[x]/(x^q - \pi_0) \to O_L : x \mapsto \pi_L$ is an isomorphism. Hence the map $O_L[x]/(x^q) \to (O_L \otimes_{O_{L_0}} O_L)^{\sim} :$ $x \mapsto 1 - \frac{1\otimes\pi_L}{\pi_L\otimes 1}$ is also an isomorphism. Let I be the kernel of the map $O_{(W\times_{T_0}W)^\sim} \to O_{(W\times_T W)^\sim}$. The product $(W \times_{T_0} W)^{\sim}$ is flat over $(T \times_{T_0} T)^{\sim}$ by Proposition 3.35. Hence we have an isomorphism $I^i/I^{i+1} \simeq O_{(W\times_T W)^\sim}$ for $0 \le i < q$ and $I^q = 0$.

1. Since the closed immersion $(W \times_T W)^{\sim} \to (W \times_{T_0} W)^{\sim}$ is a nilpotent immersion, it induces an isomorphism on the K-groups of coherent sheaves.

2. For a coherent $O_{(X\times_S X)^\sim}$-module $\mathcal{F}$, we have $[[\mathcal{F}, O_{(W\times_{T_0}W)^\sim}]] = \sum_{i=0}^{q-1}[[\mathcal{F}, I^i/I^{i+1}]] = q[[\mathcal{F}, O_{(W\times_T W)^\sim}]]$ by Lemma 2.12. Hence the assertion follows.

3. Similarly, for a coherent $O_{X_K\times_K X_K}$-module $\mathcal{F}$ and for $\sigma \in G_0$, we have

$$(\mathcal{F}, O_{W_L\times_{L_0}W_{\sigma,L}}) = \sum_{i=0}^{q-1}(\mathcal{F}, I^i O_{W_L\times_{L_0}W_{\sigma,L}}/I^{i+1}O_{W_L\times_{L_0}W_{\sigma,L}}) = q(\mathcal{F}, O_{W_L\times_L W_{\sigma,L}}).$$

Hence the assertion follows.

4. For a coherent $O_{W_L\times_L W_{\sigma,L}}$-module $\mathcal{F}$, we have $[\mathcal{F} \otimes^L_{O_{L_0}} O_{L_0}/m_{L_0}] = [\mathcal{F} \otimes^L_{O_L} O_L/m_L^q] = q[\mathcal{F} \otimes^L_{O_L} O_L/m_L]$. Thus the assertion follows.

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