The Minimal Discrepancy Coefficients of Terminal Singularities in Dimension 3

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Theorem. Let \((X, P)\) be a 3-dimensional terminal singularity of index \(r > 1\), \(\mu: Y \rightarrow X\) a resolution of singularities with exceptional divisors \(E_j (1 \leq j \leq t)\), and \(K_Y = \mu^*K_X + \sum_{j=1}^{t} d_j E_j\). Then \(\min \{d_j\} = 1/r\).

Proof. It is enough to construct a partial resolution \(\nu: X' \rightarrow X\) from a normal variety \(X'\) with an exceptional divisor \(E\) whose discrepancy coefficient is \(1/r\), i.e., \(K_X' = \nu^*K_X + \frac{1}{r}E + (\text{other components})\). We shall use the classification of terminal singularities in dimension 3 by Mori [M]. \((X, P)\) is analytically isomorphic to the singularity of a hypersurface, denoted again by \(X\), on a quotient space \(W = \mathbb{C}^4/G\) with \(G = \mathbb{Z}/(r)\) at the image of the origin of \(\mathbb{C}^4\). \(W\) is said to be of type \(\frac{1}{r}(a, b, c, d)\) if the action of a generator of \(G\) to the coordinates is given by

\[(x, y, z, w) \rightarrow (\xi^a x, \xi^b y, \xi^c z, \xi^d w)\]

for a primitive root of unity \(\xi\). \(X\) is defined by a semi-invariant \(\varphi(x, y, z, w)\). We have the following cases.

Case 1. \(W\) is of type \(\frac{1}{r}(a, -a, 0, 1)\) and \(\varphi = xy + f(z, w^r)\), where \(r\) and \(a\) are positive integers such that \(0 < a \leq r.\)
(r, a) = 1. Let \( k = \text{ord}(f) \) for \( \text{wt}(z, w) = (1, 1/r) \), and \( \sigma: W' \to W \) the weighted blow-up with weights

\[
\text{wt}(x, y, z, w) = (a/r + i, k - i - a/r, 1, 1/r)
\]

for an arbitrarily fixed \( i \) with \( 0 \leq i < k \). Then there is an affine open subset \( U \) of \( W' \) which has a quotient singularity of type

\[
\frac{1}{a + ir}(- r, (k - i)r - a, r, 1),
\]

and \( \sigma \) is given on \( U \) by

\[
(x, y, z, w) \to (x^{a/r + i}, x^{k - i - a/r}y, xz, x^{1/r}w).
\]

The strict transform \( X' \) is defined on \( U \) by \( y + f(xz, xw^r)x^{-k} = 0 \), the equation of the exceptional divisor \( F \) of \( \sigma \) is given by \( x = 0 \), and \( E = X' \cap F \) is reduced. We put \( K_{W'} = \sigma^* K_W + \alpha F \), \( \sigma^* X = X' + \beta F \) and \( K_{X'} = \nu^* K_X + dE \) for \( \nu = \sigma|_{X'} \). Then

\[
\alpha = (a/r + i) + (k - i - a/r) + 1 + 1/r - 1 = k + 1/r,
\]

\[
\beta = k, \text{ hence } d = \alpha - \beta = 1/r.
\]

In this case, one can prove that \( X' \) has only terminal singularities and \( E \) is irreducible.

**Case 2.** \( W \) is of type \( \frac{1}{2}(1, 0, 1, 1) \) and \( \varphi = x^2 + y^2 + f(z, w) \).

Let \( \text{ord}(f) = 2k \), and we consider the weighted blow-up \( \sigma: W' \to W \) with weights

\[
\text{wt}(x, y, z, w) = \begin{cases} 
(k/2, (k + 1)/2, 1/2, 1/2) & \text{if } k \text{ is odd} \\
((k + 1)/2, k/2, 1/2, 1/2) & \text{if } k \text{ is even}.
\end{cases}
\]

Then an affine open subset \( U \) of \( W \) is of type \( \frac{1}{k + 1}(k, -2, 1, 1) \) (resp. \( \frac{1}{k + 1}(-2, k, 1, 1) \)) if \( k \) is odd (resp. even), and \( \sigma \) is given by

\[
(x, y, z, w) \to \begin{cases} 
(xy^{k/2}, y^{(k+1)/2}, y^{1/2}z, y^{1/2}w) & \text{if } k \text{ is odd} \\
(x^{(k+1)/2}, x^{k/2}y, x^{1/2}z, x^{1/2}w) & \text{if } k \text{ is even}.
\end{cases}
\]

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The equation of $X'$ is $x^2 + y + f(y^{1/2}z, y^{1/2}w)y^{-k} = 0$ (resp. $x + y^2 + f(x^{1/2}z, x^{1/2}w)x^{-k} = 0$), the exceptional divisor $F$ is defined by $y = 0$ (resp. $x = 0$), and $E = X' \cap F$ is reduced. Since $\alpha = k + 1/2$ and $\beta = k$, we have $d = 1/2$.

**Case 3.** $W$ is of type $\frac{1}{2}(1, 1, 0, 1)$ and $\varphi = w^2 + f(x, y, z)$, where $\text{ord}(f) = 3$. We consider $\sigma : W' \longrightarrow W$ with $\text{wt}(x, y, z, w) = (1/2, 1/2, 1, 3/2)$. Then an open set $U$ is of type $\frac{1}{3}(1, 1, 2, -2)$, and $\sigma$ is given by $(x, y, z, w) \longrightarrow (xw^{1/2}, yw^{1/2}, zw, w^{3/2})$.

$X'$ and $F$ has equations $w + f(xw^{1/2}, yw^{1/2}, zw)w^{-2} = 0$ and $w = 0$, respectively. Thus $E$ has a reduced irreducible component $E_1$ defined by $z = w = 0$. If we put $K_{X'} = n^*K_X + dE_1 + (\text{other components})$, then $\alpha = 5/2$, $\beta = 2$, hence $d = 1/2$.

**Case 4.** $W$ is of type $\frac{1}{3}(1, 2, 2, 0)$ and $\varphi = w^2 + f(x, y, z)$, where $\text{ord}(f) = 3$. Moreover, if we write $f = f_3 + (\text{higher order terms})$, then $f_3 = x^3 + y^3 + z^3$, $x^3 + yz^2$, or $x^3 + y^3$. In the first case (resp. the remaining cases), we consider $\sigma$ with $\text{wt}(x, y, z, w) = (2/3, 1/3, 1/3, 1)$ (resp. $(2/3, 4/3, 1/3, 1)$). Then $U$ is of type $\frac{1}{3}(2, 1, 1, 0)$ and $\sigma$ is given by

$$(x, y, z, w) \longrightarrow (xw^{2/3}, yw^{1/3}, zw^{1/3}, w)$$

(resp. $$(xw^{2/3}, yw^{4/3}, zw^{1/3}, w)$$).

The equation of $X'$ is given by $w + x^3w + y^3 + z^3 + \ldots$, $1 + x^3 + yz^2 + \ldots$, or $1 + x^3 + y^3w^2 + \ldots$, while $F$ is by $w = 0$. So $F$ reduced, $\alpha = 4/3$ (resp. 7/3), $\beta = 1$ (resp. 2), and $d = 1/3$. 

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Case 5. W is of type $\frac{1}{2}(0, 1, 2, 1, 1)$ and $\varphi = w^2 + x^3 + xf(y, z) + g(y, z)$, where $\text{ord}(f) \geq 4$ and $\text{ord}(g) = 4$. We write $g = g_4 +$ (higher order terms). If $g_4$ is a square, we may assume that $g_4 = y^4$ or $y^2z^2$. If $g_4$ is not (resp. is) a square, we consider $\sigma$ with $\text{wt}(x, y, z, w) = (1, 1/2, 1/2, 3/2)$ (resp. $(1, 3/2, 1/2, 3/2)$).

Then $U$ is of type $\frac{1}{2}(0, 1, 1, 1)$ if we put

\[
(x, y, z, w) \mapsto (x, x^{1/2}y, x^{1/2}z, x^{3/2}w)
\]

(resp. $(x, x^{3/2}y, x^{1/2}z, x^{3/2}w)$).

$X'$ is given by $xw^2 + x + f(x^{1/2}y, x^{1/2}z)x^{-1} + g(x^{1/2}y, x^{1/2}z)x^{-2} = 0$ (resp. $w^2 + 1 + f(x^{3/2}y, x^{1/2}z)x^{-2} + g(x^{3/2}y, x^{1/2}z)x^{-3} = 0$), and $F$ is by $x = 0$. So $F$ is reduced, $\alpha = 5/2$ (resp. $7/2$), $\beta = 2$ (resp. 3), and $d = 1/2$.

Case 6. W is of type $\frac{1}{4}(1, 3, 2, 1)$ and $\varphi = x^2 + y^2 + f(z, w^2)$.
If $\text{ord}(f) = k$ for $\text{wt}(z, w) = (1, 1/2)$, we take $\sigma$ with

\[
\text{wt}(x, y, z, w) = \begin{cases} 
(k/4, (k + 2)/4, 1/2, 1/4) & \text{if } k \equiv 1 \pmod{4} \\
((k + 2)/4, k/4, 1/2, 1/4) & \text{if } k \equiv 3 \pmod{4}.
\end{cases}
\]

Then $U$ is of type $\frac{1}{k + 2}(k, -4, 2, 1)$ (resp. $\frac{1}{k + 2}(-4, k, 2, 1)$) if $k \equiv 1$ (resp. 3) (mod 4), and $\sigma$ is given by

\[
(x, y, z, w) \mapsto \begin{cases} 
(xy^{k/4}, y^{(k+2)/4}, y^{1/2}z, y^{1/4}w) & \text{if } k \equiv 1 \pmod{4} \\
(x^{(k+2)/4}, x^{k/4}y, x^{1/2}z, x^{1/4}w) & \text{if } k \equiv 3 \pmod{4}.
\end{cases}
\]

$X'$ is given by $x^2 + y + f(y^{1/2}z, y^{1/2}w^2)y^{-k/2} = 0$ (resp. $x + y^2 + f(x^{1/2}z, x^{1/2}w^2)x^{-k/2} = 0$) and $F$ by $y = 0$ (resp. $x = 0$) if $k \equiv 1$ (resp. 3) (mod 4). Thus $E$ is reduced, $\alpha = (2k + 1)/4$ and $\beta =$

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$k/2$, hence $d = 1/4$. Q.E.D.

References