TERMINATION OF LOG FLIPS FOR ALGEBRAIC 3-FOLDS

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1. Introduction

We shall prove that there exists no infinite sequence of successive log flips for algebraic 3-folds.

Let $X$ be a normal $\mathbb{Q}$-factorial variety and $B$ a $\mathbb{Q}$-divisor such that the pair $(X, B)$ has only weak log terminal singularities (cf. [3] or Sec. 2 for the terminology). A log flip for $(X, B)$ is a diagram

$$
X \xrightarrow{\phi} Y \xleftarrow{\psi} X^+
$$

consisting of projective birational morphisms between normal varieties such that

(1) $\rho(X/Y) = 1$, i.e., $\phi$ is not an isomorphism and for any two curves $C$ and $C'$ which are mapped to points by $\phi$, there exists a positive number $\alpha$ such that $C \sim_{\text{num}} \alpha C'$,

(2) $\rho(X^+/Y) = 1$,

(3) $\text{codim Exc}(\phi) \geq 2$, where $\text{Exc}$ denotes the exceptional locus,

(4) $\text{codim Exc}(\phi^+ \psi) \geq 2$,

(5) $-K_X + B$ is $\phi$-ample,

(6) $K_{X^+} + B^+$ is $\phi^+$-ample, where $B^+$ is the strict transform of $B$ on $X^+$.

The resulting pair $(X^+, B^+)$ is automatically $\mathbb{Q}$-factorial and weak log terminal. The main result of this paper is the following:

**Theorem 1.** Let $X$ be a 3-dimensional normal $\mathbb{Q}$-factorial variety and $B$ a $\mathbb{Q}$-divisor such that the pair $(X, B)$ has only weak log terminal singularities. Then there exists no infinite sequence of successive log flips such as

$$
X = X_0 \xrightarrow{\phi_0} Y_0 \xleftarrow{\psi_0} X_1 \xrightarrow{\phi_1} Y_1 \xleftarrow{\psi_1} X_2 \xrightarrow{\phi_2} Y_2 \xleftarrow{\psi_2} X_3 \xrightarrow{\phi_3} \ldots
$$

where the first pair of arrows is a log flip for $(X_0, B_0)$ with $B_0 = B$, the second is for $(X_1, B_1)$ with the strict transform $B_1$ of $B_0$, the third is for $(X_2, B_2)$ with the strict transform $B_2$ of $B_1$, and so on.

Shokurov [6] proved the following results:

(i) (Existence) For $(X, B)$ as above with $\dim X = 3$ and for a projective birational morphism $\phi : X \to Y$ satisfying conditions (1), (3) and (5) above, there exists $\phi^+ : X^+ \to Y$ completing the log flip diagram.

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(ii) (Special Termination) There exists no infinite sequence as in Theorem 1 which satisfies \( \text{Exc}(\varphi_n) \cap [B_n] \neq \emptyset \) for infinitely many \( n \).

So it is enough to prove Theorem 1 under the additional assumption that \( \text{Exc}(\varphi_n) \cap [B_n] = \emptyset \) for all \( n \). By replacing \( X \) by \( X \setminus [B] \), we may assume that \( [B] = 0 \), hence \((X, B)\) has only log terminal singularities.

The method of our proof is a combination of the use of the difficulty in [5] modified in [4] together with an inductive argument in [1]. The most important step is to find a universal constant \( q \in \mathbb{Z}_{>0} \) such that all the prime divisors on the \( X_n \) are \( q \)-Cartier (Lemma 7).

By the Log Minimal Model Program ([3]), the above results combined yield the following existence theorem of log minimal models. We note that, even in the case in which \( b_i = 1 \) for all \( i \), we need our new result, since log terminal singularities with \( [B] = 0 \) may appear after divisorial contractions.

**Theorem 2.** Let \( X_0 \) be a 3-dimensional nonsingular projective variety and \( B_0 = \sum b_iS_i \) a \( \mathbb{Q} \)-divisor such that \( 0 < b_i \leq 1 \) and the \( S_i \) are mutually distinct nonsingular prime divisors with normal crossings. Then there exist a pair \((X, B)\) of normal projective \( \mathbb{Q} \)-factorial variety and a \( \mathbb{Q} \)-divisor having only weak log terminal singularities, and a birational map \( f : X_0 \dashrightarrow X \) which is surjective in codimension 1, i.e., the image of the domain of \( f \) contains all the points of codimension 1, such that one of the following holds: \( B \) being the image of \( B_0 \),

1. there exists a surjective morphism \( \varphi : X \to Y \) with connected fibers to a normal projective variety \( Y \) such that \( \dim Y < \dim X \) and \( -(K_X + B) \) is \( \varphi \)-ample,
2. \( K_X + B \) is nef.

2. **Terminal Blowing-Up**

Let \( X \) be a normal \( \mathbb{Q} \)-factorial variety and \( B = \sum_{i=1}^N b_iS_i \), a \( \mathbb{Q} \)-divisor, where \( 0 < b_i \leq 1 \) and the \( S_i \) are mutually distinct prime divisors. The pair \((X, B)\) is said to have only weak log terminal singularities (or to be weak log terminal) if there exists a projective birational morphism \( \mu : M \to X \) from a nonsingular variety \( M \) with a normal crossing divisor \( F = \sum F_j \) such that

1. \( \mu : M \setminus F \to X \setminus (\text{Sing}(X) \cup \text{Supp}(B)) \), so the \( F_j \) are the exceptional divisors for \( \mu \) and the strict transforms of the \( S_i \),
2. \( K_M = \mu^*(K_X + B) + \sum a_jF_j \) with \( a_j > -1 \) for all \( F_j \) which are exceptional for \( \mu \).

Moreover, if \( b_i < 1 \) for all \( i \), then \((X, B)\) is called log terminal. In this case, the above condition (2) also holds when \( \mu \) is replaced by any other resolution. In particular, there exists a \( \mu \) for which the \( F_j \) with \( a_j < 0 \) are disjoint. For such a resolution, we define the number

\[
e(X, B) = \# \{ j | F_j \text{ is exceptional for } \mu \text{ and } a_j \leq 0 \}.
\]

This number is well defined since there is no \( j \) with \( a_j = -1 \).
The discrepancy coefficient \( a_j = a(F_j) \) is determined by the discrete valuation \( v_j \) of \( C(X) \) associated with the prime divisor \( F_j \), and independent of the resolution \( \mu \). In this sense, we can define the discrepancy coefficient \( a(v) \) for any discrete valuation \( v \) of \( C(X) \) by \( a(v) = a(F) \) if the center of \( v \) on some resolution \( M \) is a prime divisor \( F \). If we have to specify \((X, B)\), we write \( a(X, B; v) \) instead of \( a(v) \). \( v \) is called exceptional if its center on \( X \) is not a prime divisor. Then we have

\[
e(X, B) = \#\{v|v \text{ is exceptional and } a(v) \leq 0\}.
\]

The following lemma is fundamental in the proof of the termination of flips or log flips.

**Lemma 3.** ([5], [3, 5.1.11]) Let \((X, B)\) be a \( \mathbb{Q} \)-factorial weak log terminal pair, \( X \xrightarrow{\sigma} Y \xrightarrow{\varphi} X^+ \) a log flip for \((X, B)\), and \( v \) a discrete valuation of \( C(X) \). Then

\[
a(X, B; v) \leq a(X^+, B^+; v)
\]

and the strict inequality holds if and only if the center of \( v \) on \( X \) is contained in \( \text{Exc}(\varphi) \).

Note that the same conclusions as in Lemma 3 hold for divisorial contractions. 

\((X, B)\) is called terminal if \( e(X, B) = 0 \). In the case \( B \neq 0 \), we assume that \( b_1 \geq \cdots \geq b_N \). In this case, we define \( d(X, B; m_1, \ldots, m_N) = \#\{v|v \text{ is exceptional and } a(v) < 1 - \sum m_i b_i\} \) for nonnegative integers \( m_i \) with \( m_1 > 0 \), and

\[
d(X, B) = \sum_{m_1, \ldots, m_N} d(X, B; m_1, \ldots, m_N).
\]

Note that all the entries of the sum are finite and are 0 except a finite number of the \((m_1, \ldots, m_N)\). If \( B = 0 \), we set simply

\[
d(X) = \#\{v|a(v) < 1\}
\]

which is the difficulty of \( X \) [5].

**Lemma 4.** Theorem 1 holds under the additional assumption that \( e(X, B) = 0 \).

**Proof.** Let \( S_1^n \) be the strict transforms of the \( S_i \) on \( X_n \). We may assume that \( B = 0 \) or \( \text{Exc}(\varphi_n) \cap S_1^n \neq \emptyset \) for infinitely many \( n \), since we can otherwise replace \( X_n \) by \( X_n \setminus S_1^n \) for \( n \geq n_0 \) with some \( n_0 \). In the latter case, if \( \text{Exc}(\varphi_n) \cap S_1^n \neq \emptyset \) but no prime component \( C^+ \) of \( \text{Exc}(\varphi_n) \) is contained in \( S_1^{n+1} \), then \( (S_1^n \cdot C^+) > 0 \) for any \( C^+ \), hence \( (S_1^n \cdot C) < 0 \) and \( C \subset S_1^n \) for any prime component \( C \) of \( \text{Exc}(\varphi_n) \). Since \( S_1 \) cannot be contracted infinitely many times, there is a prime component \( C^+ \) of \( \text{Exc}(\varphi_1) \) contained in \( S_1^{n+1} \) for some \( n \). Since \( X_{n+1} \) is terminal, it is generically nonsingular along \( C^+ \). Let \( v \) be the discrete valuation of \( C(X) \) given by the order of zeroes at the generic point of \( C^+ \), and \( m_i \) the multiplicities of the \( S_i^{n+1} \) along \( C^+ \). Then

\[
a(X_{n+1}, B_{n+1}; v) = 1 - \sum m_i b_i.
\]
Since $a(X_n, B_n; v) < a(X_{n+1}, B_{n+1}; v)$ by Lemma 3, we have $d(X_n, B_n) > d(X_{n+1}, B_{n+1})$ if $B \neq 0$. If $B = 0$, then we have $d(X_n) > d(X_{n+1})$ for any $n$. Therefore, an infinite sequence of log flips is impossible.

The following theorem asserts the existence of a terminal blowing-up for a log terminal pair.

**Theorem 5.** Let $(X, B)$ be a 3-dimensional log terminal pair. Then there exist a $\mathbb{Q}$-factorial terminal pair $(V, B_V)$ and a projective birational morphism $\mu : V \rightarrow X$ such that

1. $\mu_\ast B_V = B$,
2. $K_V + B_V = \mu_\ast (K_X + B)$,
3. the number of exceptional divisors of $\mu$ is equal to $e(X, B)$.

**Proof.** Let $\mu_0 : V_0 \rightarrow X$ be a projective birational morphism from a nonsingular variety $V_0$ with a normal crossing divisor $F = \sum F_j$ such that $K_{V_0} = \mu_0^\ast (K_X + B) + \sum a_j F_j$ with $a_j > -1$ for all $j$. We take $\mu_0$ so that the $F_j$ with $a_j < 0$ are disjoint. Let $B_0 = \sum_{a_j < 0} |a_j| F_j$. Then $\mu_0^\ast B_0 = B$.

We apply the Log Minimal Model Program to the pair $(V_0, B_0)$ over $X$ ([13]). We construct a sequence of $\mathbb{Q}$-factorial terminal pairs $(V_n, B_n)$ for $n = 0, 1, \ldots$ with projective birational morphisms $\mu_n : V_n \rightarrow X$ by divisorial contractions and log flips. We have to prove the following:

(i) if $(V_n, B_n)$ is terminal, $\varphi : V_n \rightarrow V_{n+1}$ is a divisorial contraction with respect to $K_{V_n} + B_n$ over $X$, and if $E$ is the exceptional divisor, then the discrepancy coefficient $a(X, B; E) > 0$, hence $E$ is not contained in the support of $B_n$, and the pair $(V_{n+1}, B_{n+1})$ is again terminal,

(ii) if $K_{V_n} + B_n$ is $\mu_n$-nef, then $K_{V_n} + B_n = \mu_n^\ast (K_X + B)$.

Then by Lemma 4, the sequence terminates after a finite number of steps, and we obtain the desired terminal blowing-up.

First, we prove (i). Let $U$ be an affine open subset of $X$ containing the generic point of $\mu_n(E)$. In the case in which $\dim \mu_n(F) = 0$ (resp. 1), let $H$ be a general member of a very ample linear system on $\mu_n^{-1}(U)$ (resp. of the pull back by $\mu_n$ of a very ample linear system on $U$). Let $C = E \cap H$. By construction, we can write $K_{V_n} + B_n = \mu_n^\ast (K_X + B) + \Delta$ for an effective exceptional divisor $\Delta$. Since $(\Delta \cdot C) < 0$, $E$ is in the support of $\Delta$, hence $a(E) > 0$.

If $\Delta$ is $\mu_n$-nef, then $\Delta = 0$ by the Hodge index theorem applied on $H$, hence (ii).

3. **Proof of Theorem 1**

We shall prove Theorem 1 and the following two lemmas together by induction in 3 steps.

**Lemma 6.** Let $(X, B)$ be a $\mathbb{Q}$-factorial log terminal pair which is not terminal. Then there exist a $\mathbb{Q}$-factorial log terminal pair $(V, B_V)$ and a projective birational morphism $\mu : V \rightarrow X$ such that

1. $\mu_\ast B_V = B$,
(2) $K_V + B_V = \mu^*(K_X + B)$,
(3) the exceptional locus of $\mu$ is a prime divisor and $e(V, B_V) = e(X, B) - 1$.

**Lemma 7.** Let $e$ be a nonnegative integer and $c$ a positive rational number. Then there exists a positive integer $q$ which satisfies the following condition: if $(X, B)$ is a $\mathbb{Q}$-factorial log terminal pair with $e(X, B) = e$ and such that $a(X, B; v) \geq c$ for any discrete valuation $v$ of $\mathbb{C}(X)$ with $a(X, B; v) > 0$, and if $D$ is a prime divisor on $X$, then $qD$ is a Cartier divisor.

If $e(X, B) = 0$, Theorem 1 is proved in Lemma 4, and Lemma 6 is trivial.

**Step 0.** Lemma 7 for $e(X, B) = 0$.

**Proof.** Let $(X, B)$ be a $\mathbb{Q}$-factorial terminal pair such that $a(X, B; v) \geq c$ for any exceptional discrete valuation $v$ of $\mathbb{C}(X)$. Let $P$ be any singular point of $X$, and $r$ its index of singularity, i.e., $r$ is the minimum positive integer such that $rK_X$ is a Cartier divisor at $P$. By [2], there exists an exceptional discrete valuation $v$ such that $a(X, B; v) \leq 1/r$, hence $c \leq 1/r$. By [1, 5.2], $rD$ is a Cartier divisor at $P$ for any prime divisor $D$ on $X$. Thus $q = [1/c]$! is enough.

**Step 1.** For a positive integer $e$, Theorem 1 for all pairs $(X, B)$ such that $e(X, B) \leq e - 2$ implies Lemma 6 for all $(X, B)$ such that $e(X, B) = e$.

**Proof.** Let $(X, B)$ be a $\mathbb{Q}$-factorial log terminal pair such that $(X, B) = e$, and $\mu_0 : V_0 \to X$ the terminal blowing-up from a terminal pair $(V_0, B_0)$ obtained in Theorem 5. If $e = 1$, then $(V, B_V) = (V_0, B_0)$ works. Otherwise, let us pick an exceptional divisor $E_1$ of $\mu_0$, and apply the Log Minimal Model Program to the pair $(V_0, B_0 + E_1)$ over $X$ for a sufficiently small positive rational number $\varepsilon$. By Theorem 1 for the case $e(X, B) = 0$, after a finite number of log flips, we have a divisorial contraction to obtain a pair $(V_1, B_1)$ with a projective birational morphism $\mu_1 : V_1 \to X$ such that $e(V_1, B_1) = 1$, $\mu_1^* B_1 = B_1$, and $K_{V_1} + B_1 = \mu_1^*(K_X + B)$. If $e = 2$, then $(V, B_V) = (V_1, B_1)$. Otherwise, take an exceptional divisor $E_2$ of $\mu_1$, and apply the Log MMP to $(V_1, B_1 + E_2)$, and so on.

**Step 2.** For a positive integer $e$, Lemma 6 for any $\mathbb{Q}$-factorial log terminal pair $(X, B)$ such that $e(X, B) = e$, and Lemma 7 such that $e(X, B) = e - 1$ imply Lemma 7 in the case $e(X, B) = e$.

**Proof.** Let us fix a positive number $c$. Let $q_1$ be the positive integer corresponding to $e - 1$ and $c$ given by Lemma 7. Let $(X, B)$ be a $\mathbb{Q}$-factorial log terminal pair such that $(X, B) = e$ and $a(X, B; v) \geq c$ for any discrete valuation $v$ of $\mathbb{C}(X)$ with $a(X, B; v) > 0$. Let $\mu : V \to X$ be the projective birational morphism from a $\mathbb{Q}$-factorial log terminal pair $(V, B_V)$ with $e(V, B_V) = e - 1$ obtained in Lemma 6. Let $E$ be the exceptional divisor of $\mu$, and let $E' \to E$ be the minimal resolution. Let $-b = a(X, B; E)$. By assumption, we have $1 - b \geq c$.

Since $(V, (b + \varepsilon)E)$ is log terminal for a small positive rational number $\varepsilon$ and $- (K_V + (b + \varepsilon)E)$ is $\mu$-ample, it follows that $E'$ is covered by rational curves. So there
is a movable rational curve $C$ on $E$ such that $\mu(C)$ is a point and $-3 \leq (K_E, C) \leq (K_V + E). C < 0$ for the strict transform $C'$ by [3, 5.1.9] (cf. [1, 6.8]).

Let $D$ be a prime divisor on $X$ and $D'$ the strict transform on $V$. Since $q_1E$ and $q_1D'$ are Cartier divisors, $s = (-q_1E, C)$ and $t = (q_1D', C)$ are integers. Since $(K_V + B_V). C = 0$, we have $0 < (1 - b)s \leq 3q_1$. Since $(sD' + tE). C = 0$, we have $s\mu^*D = sD' + tE$ and $sq_1D$ is a Cartier divisor ([3, 3.2.5(2)]). Thus $q = \lfloor 3q_1/(1 - b) \rfloor! q_1$ works.

**Step 3.** For a positive integer $e$, Lemma 7 for any $\mathbb{Q}$-factorial log terminal pair $(X, B)$ such that $e(X, B) = e$ and Theorem 1 such that $e(X, B) < e$ imply Theorem 1 such that $e(X, B) = e$.

**Proof.** Let $(X, B)$ be a $\mathbb{Q}$-factorial log terminal pair with $e(X, B) = e$, and suppose that there exists an infinite sequence of log flips starting from $(X, B)$ as in Theorem 1. If $e(X_n, B_n) < e$ for some $e$, then the sequence terminates by the assumption, so we assume that $e(X_n, B_n) = e$ for all $n$ in the following. Then by Lemma 3, there is a positive number $c$ such that $a(X_n, B_n; v) \geq c$ if $v > 0$ for any $n$ and any discrete valuation $v$ of $C(X)$. Thus there exists a positive integer $q$ such that $qD$ is a Cartier divisor for any prime divisor $D$ on any $X_n$. By replacing $q$ by its multiple with the denominators of the $b_i$, the coefficients of $B$, we may assume that $q(K_{X_n} + B_n)$ is a Cartier divisor for any $n$, hence $a(X_n, B_n; v) \in (1/q)\mathbb{Z}$ for any $v$.

Let $v_1, \ldots, v_e$ be discrete valuations of $C(X)$ such that $a(X, B; v_j) \leq 0$. If there is a $j$ and $n$ such that the center $C_{jn}$ of $v_j$ on $X_n$ is contained in $\text{Exc}(\phi_n)$, then $\sum_{j=1}^e a(X_n, B_n; v_j)$ increases by Lemma 3. Then it cannot be repeated infinitely many times, so we may assume that the $C_{jn}$ are not contained in the $\text{Exc}(\phi_n)$ for any $j$ and $n$. Thus $|a(X_n, B_n; v_j)| = c_j$ is constant on $n$. Suppose that $c_1 \geq \cdots \geq c_e$. We may assume that there are infinitely many $n$ such that $C_{1n} \cap \text{Exc}(\phi_n) \neq \emptyset$, because we can otherwise replace $X_n$ by $X_n \setminus C_{1n}$ for $n \geq n_0$ with some $n_0$, and reduce it to the case in which $e(X, B) = e - 1$.

As in the proof of Lemma 4, we have $B = 0$ or there are prime components $C^+_n$ of $\text{Exc}(\phi^+_n)$ contained in $S^{*+1}_n$ for infinitely many $n$. Let $b_1$ be the maximum of the $b_i$ if $B \neq 0$. We set $b_1 = 0$ if $B = 0$. Since there are only a finite number of discrete valuations $v$ of $C(X)$ such that $a(X, B; v) < 1 - \max\{b_1, c_1\}$, we can define

$$\Delta_n = \{v | a(X_n, B_n; v) < 1 - \max\{b_1, c_1\}\}$$

$$d_n = \# \Delta_n$$

$$\sigma_n = \sum_{v \in \Delta_n} a(X_n, B_n; v).$$

In the case in which $b_1 \geq c_1$, if $C^+_n$ is contained in $S^{*+1}_n$, then $d_n > d_{n+1}$ or $\sigma_n < \sigma_{n+1}$ as in the proof of Lemma 4. Since $\sigma_n \in (1/q)\mathbb{Z}$, $d_n$ will decrease eventually also in the latter case. Therefore, the infinite sequence of log flips is impossible in this case.
In the case in which \( b_1 < c_1 \), if \( C_{1,n} \cap \text{Exc}(\varphi_n) \neq 0 \), let \( \mu : V \to X_{n+1} \) be the terminal blowing-up obtained in Theorem 5, \( E_1 \) the exceptional divisor corresponding to \( v_1 \), and let \( v_0 \) be the discrete valuation given by the order of zeroes along a prime component of a fiber of \( E_1 \to \mu(E_1) = C_{1,n+1} \) over a point in \( C_{1,n+1} \cap \text{Exc}(\varphi_n^+) \). Then we have \( a(X_{n+1}, B_{n+1}; v_0) \leq 1 - c_1 \), hence \( d_n > d_{n+1} \) or \( \sigma_n < \sigma_{n+1} \) also in this case, thus we are done.

References
