On genus two Heegaard splittings of some non-simple 3-manifolds

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1. Main Theorem 1

Definition M : an orientable closed 3-manifold.
$(V_1, V_2; F)$, $(W_1, W_2; G)$: two Heegaard splittings of the same genus
$(V_1, V_2; F)$ and $(W_1, W_2; G)$ are **homeomorphic** (**isotopic** resp.)
$\Longleftrightarrow \exists$ homeomorphism $f : M \longrightarrow M$ (isotopy $f_t : M \longrightarrow M$ resp.)
s.t. $f(F) = G$ ($f_1(F) = G$ resp.).

Theorem 1 (Morimoto)
S_1, S_2 : Seifert fibered spaces over a disk with 2 exceptional fibers
$f : \partial S_2 \longrightarrow \partial S_1$ homeomorphism
$\Longrightarrow M = S_1 \cup_f S_2$ admits at most 4 non-isotopic Heegaard splittings of genus two.

Remark (1) Morimoto listed up all possible Heegaard splittings up to isotopy.
(2) If $M = S_1 \cup_f S_2$ admits four non-isotopic Heegaard splittings of genus two, then

$$S_i = S^3 \setminus T_{2,2n_i+1} \text{ and } f : h_2 \longmapsto \varepsilon m_1, m_2 \longmapsto \delta h_1 \qquad (*)$$

,where $n_i > 1, n_i \in \mathbb{N}$, $\varepsilon\delta = \pm 1$ and $T_{2,2n_i+1}$ is a torus knot of type $(2, 2n_i + 1)$.

Main theorem 1
Any two Heegaard splittings on Morimoto's list are not isotopic.
In particular, if $(*)$ holds,
then $M = S_1 \cup_f S_2$ admits exactly 4 non-isotopic Heegaard splittings.

The homeomorphism classification of Heegaard splittings of M can be obtained from Main Theorem 1 and by calculating the mapping class group of M. For example, if $(*)$ holds, then

(1) M admits exactly 4 Heegaard splittings up to homeomorphism when $n_1 \neq n_2$,
(2) M admits exactly 3 Heegaard splittings up to homeomorphism when $n_1 = n_2$.

2. Generalization to Other Non-simple 3-manifolds

By arguments similar to those for Theorem 1 and Main Theorem 1, we can classify Heegaard splittings of some family of manifolds containing essential separating torus (cf. Kobayashi '84). We give an example.

Main theorem 2

$M_1 = S(D^2; \beta_1/\alpha_1, \beta_2/\alpha_2)$

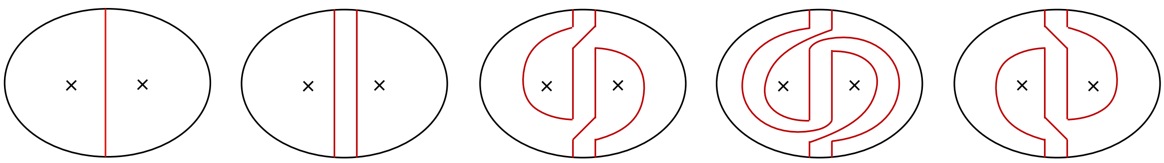
$M_2 = S^3 \setminus N(S(\alpha, \beta))$, where $S(\alpha, \beta)$ is a hyperbolic 2-bridge knot

Let $M = M_1 \cup_f M_2$ (the regular fiber of $M_1 \leftrightarrow$ the meridian loop of M_2 by f.

Then

(1) any Heegaard surface of M is isotopic to the surface obtained from one of the following surfaces by applying Dehn twists D_l along the attaching torus in the direction of a longitude l of the 2-bridge knot.

F_1 is the union of A in M_1 and the twice-punctured torus in M_2 associated with (τ_1, ρ_2),
F_2 is the union of A in M_1 and the twice-punctured torus in M_2 associated with (τ_1, ρ_2'),
F_3 is the union of A in M_1 and the twice-punctured torus in M_2 associated with (τ_2, ρ_1),
F_4 is the union of A in M_1 and the twice-punctured torus in M_2 associated with (τ_3, ρ_1'),
F_5 is the union of A in M_1 and the twice-punctured torus in M_2 associated with (ρ_2, τ_1),
F_6 is the union of A in M_1 and the twice-punctured torus in M_2 associated with (ρ_2', τ_1),
F_7 is the union of A in M_1 and the twice-punctured torus in M_2 associated with (ρ_1, τ_2),
F_8 is the union of A in M_1 and the twice-punctured torus in M_2 associated with (ρ_1', τ_2),
F_9 is the union of A_1' in M_1 and the two-bridge sphere of M_2,
F_{10} is the union of A_2' in M_1 and the two-bridge sphere of M_2,
F_{11} is the union of A_3' in M_1 and the two-bridge sphere of M_2,
F_{12} is the union of A_4' in M_1 and the two-bridge sphere of M_2.



A $\quad A'_1 \quad A'_2 \quad A'_3 \quad A'_4$

(2) The following tables give the isotopy and homeomorphism classification of the Heegaard surfaces in (1).

<table>
<tr><td colspan="2"></td><td>F_1</td><td>F_2</td><td>F_3</td><td>F_4</td><td>F_5</td><td>F_6</td><td>F_7</td><td>F_8</td><td>F_9</td><td>F_{10}</td><td>F_{11}</td><td>F_{12}</td></tr>
<tr><td colspan="2">$\alpha = 5$</td><td colspan="2">○</td><td colspan="2">○</td><td colspan="2">○</td><td colspan="2">○</td><td>○</td><td>○</td><td>○</td><td>○</td></tr>
<tr><td rowspan="3">$\alpha \geq 7$</td><td>$\beta \equiv \pm 2 \pmod{\alpha}$</td><td colspan="2">○</td><td>○</td><td>○</td><td colspan="2">○</td><td>○</td><td>○</td><td>○</td><td>○</td><td>○</td><td>○</td></tr>
<tr><td>$\beta \equiv \pm 2^{-1} \pmod{\alpha}$</td><td>○</td><td>○</td><td colspan="2">○</td><td>○</td><td>○</td><td colspan="2">○</td><td>○</td><td>○</td><td>○</td><td>○</td></tr>
<tr><td>otherwise</td><td>○</td><td>○</td><td>○</td><td>○</td><td>○</td><td>○</td><td>○</td><td>○</td><td>○</td><td>○</td><td>○</td><td>○</td></tr>
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Table 1: isotopy classification

○ $\longleftrightarrow$ a family of infinitely many Heegaard surfaces obtained from F_i for the corresponding i by applying Dehn twist along the attaching torus in the direction of a longitude of the 2-bridge knot.

	$F_1 \cong F_2 \cong F_3 \cong F_4$	$F_5 \cong F_6 \cong F_7 \cong F_8$	F_9	F_{10}	F_{11}	F_{12}
$\alpha = 5$	1		1	1	1	1
$\alpha \geq 7$	1	1	1	1	1	1

Table 2-1: homeomorphism classification when $\beta^2 \equiv \pm 1 \pmod{\alpha}$

	$F_1 \cong F_2$	$F_5 \cong F_6$	$F_3 \cong F_4$	$F_7 \cong F_8$	F_9	F_{10}	F_{11}	F_{12}
$\beta \equiv \pm 2 \pmod{\alpha}$	1		1	1	1	1	1	1
$\beta \equiv \pm 2^{-1} \pmod{\alpha}$	1	1	1		1	1	1	1
otherwise	1	1	1	1	1	1	1	1

Table 2-2: homeomorphism classification when $\beta^2 \not\equiv \pm 1 \pmod{\alpha}$

3. 3-bridge Presentations

By considering double branched covering, genus-two Heegaard splittings correspond to 3-bridge presentations of 3-bridge knots or links in S^3.

Main theorem 3

There exist 3-bridge links each of which admits infinitely many 3-bridge presentations.

In fact, there exist infinitely many 3-bridge links with this property. For example,

$M_1 := S(A; (\alpha_1, \beta_1), (\alpha_2, \beta_2))$, where A is an annulus
M_2 : be the link exterior of a hyperbolic 2-bridge link $L = K_1 \cup K_2$
$M := M_1 \cup_f M_2$, where f is a homeomorphism from $\partial M_1 = A_1 \cup A_2$ to $\partial N(K_1) \cup \partial N(K_2)$
the regular fiber on $A_i \leftrightarrow$ the meridian of K_i by f $(i = 0, 1)$
$\Rightarrow$ M admits infinitely many Heegaard splittings up to isotopy.

Moreover, from each of those Heegaard splittings, we obtain a 3-bridge link which admits infinitely many 3-bridge presentations.

Example Let $M_1 = S^3 \setminus N(K_{4,10})$ and $M_2 = S^3 \setminus N(S(3, 10))$, then we obtain the following 3-bridge link with infinitely many 3-bridge presentations.