On the signature cocycle and related invariants of 3-manifolds

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Secondary invariants

signature cocycle

Meyer's function

periodic $\swarrow$ $\downarrow$ $\mathbf{Z}$ cov $\searrow$ Torelli group

η-invariant | von Neumann ρ-inv | Morita's homomorphism

signature op | bdd coh class | Casson inv

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- Signature cocycle and Meyer's function
- Eta-invariant
- von Neumann rho-invariant
- Casson invariant
- Bounded cohomology

§ Signature cocycle

Σ_g : an oriented closed C^∞-surface of genus g

$\mathcal{M}_g = \pi_0 \mathrm{Diff}_+ \Sigma_g$ mapping class group

Fix a symplectic basis of $H_1(\Sigma_g, \mathbf{Z})$

$$r : \mathcal{M}_g \to \mathrm{Sp}(2g, \mathbf{Z}) \quad \text{homology rep.}$$

$\mathcal{I}_g = \mathrm{Ker}\ r$ Torelli group

$\star$ Meyer's signature cocycle $\tau \in Z^2(\mathrm{Sp}(2g, \mathbf{Z}), \mathbf{Z})$

$A, B \in \mathrm{Sp}(2g, \mathbf{Z})$, I : the identity matrix

Define $V_{A,B} \subset \mathbf{R}^{2g} \times \mathbf{R}^{2g}$ to be

$$V_{A,B} = \left\{(x, y) \mid (A^{-1} - I)x + (B - I)y = 0\right\}$$

Define the pairing map on $\mathbf{R}^{2g} \times \mathbf{R}^{2g}$ by

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{A,B} = (x_1 + y_1) \cdot J(I - B)y_2,$$

where $\cdot$ is the inner product in $\mathbf{R}^{2g}$, $J = \left(\begin{smallmatrix} O & I \\ -I & O \end{smallmatrix}\right)$

$\Rightarrow$ Symmetric bilinear form $\underline{\text{on } V_{A,B}}$

Define

$$\tau(A, B) = \mathrm{Sign}\,(V_{A,B}, \langle\ ,\ \rangle_{A,B})$$

From Novikov additivity, $\tau(A, B)$ satisfies the cocycle condition, i.e.

$$\tau(A, B) + \tau(AB, C) = \tau(A, BC) + \tau(B, C)$$

$\Rightarrow\ \ \tau \in Z^2(\mathrm{Sp}(2g, \mathbf{Z}), \mathbf{Z})$ signature cocycle

Remark

- We can regard τ as a 2-cocycle of $\mathcal{M}_g$ by r

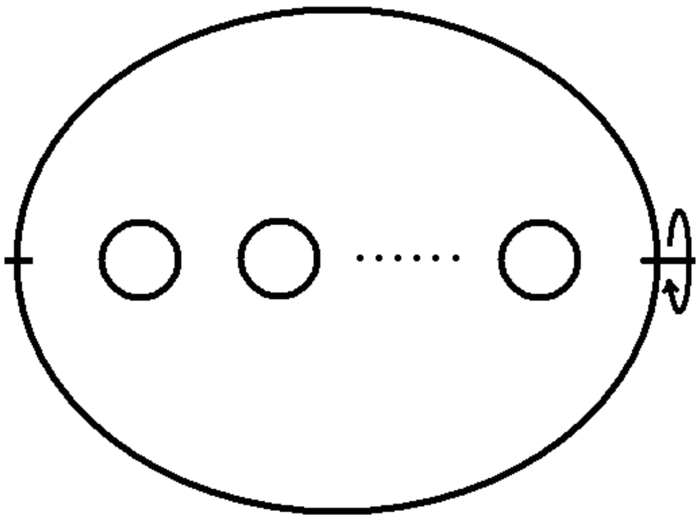
- $\tau(A,B) = \mathrm{Sign}\begin{pmatrix} W^4 \\ \downarrow \Sigma_g \\ P^2 \end{pmatrix}$, P is the pair of pants

  $\partial P = M_A \cup M_B \cup -M_{AB}$ mapping tori

- By definition, τ is a bounded 2-cocycle

  (i.e. $|\tau| \leq 2g$)

Hyperelliptic mapping class group



ι : hyperelliptic involution

$$\Delta_g = \{f \in \mathcal{M}_g \mid f\iota = \iota f\}$$

If $g = 1, 2 \quad \Rightarrow \quad \Delta_g = \mathcal{M}_g$

Fact $H^*(\Delta_g, \mathbf{Q}) = 0,\ * = 1, 2$ Cohen, Kawazumi

Hence $[\tau]$ has a finite order in $H^2(\Delta_g, \mathbf{Z})$

Fact $(2g+1)\tau \in B^2(\Delta_g, \mathbf{Z})$

$\Rightarrow$ there exists the uniquely defined mapping

$$\phi : \Delta_g \to \frac{1}{2g+1}\mathbf{Z} = \left\{ \frac{m}{2g+1} \in \mathbf{Q} \mid m \in \mathbf{Z} \right\}$$

s.t. $\delta\phi = \tau|_{\Delta_g}$ Meyer's function of Δ_g

Remark

- ϕ is a class function of Δ_g

  i.e. $\phi(hfh^{-1}) = \phi(f),\ f, h \in \Delta_g$

  $\Rightarrow$ an invariant of surface bundles over the circle

- $\delta\phi = \tau|_{\Delta_g}$ implies ϕ is a homomorphism on the Torelli group $\mathcal{I}_g \cap \Delta_g\ (g \geq 2)$

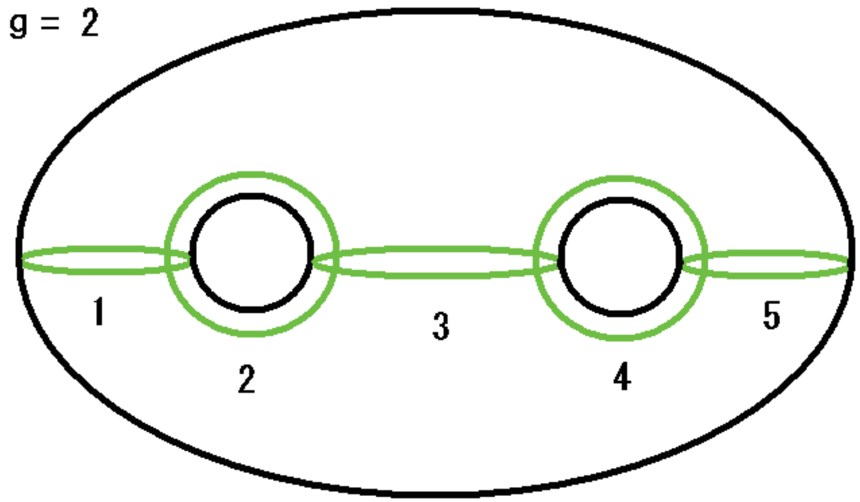
Example a presentation of Δ_g Birman-Hilden

generator : $\zeta_i \ (1 \leq i \leq 2g+1)$

relation :

$$\zeta_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \zeta_{i+1}$$
$$\zeta_i \zeta_j = \zeta_j \zeta_i \ (|i-j| \geq 2)$$
$$(\zeta_1 \cdots \zeta_{2g+1})^{2g+2} = 1$$
$$(\zeta_1 \cdots \zeta_{2g+1}^2 \cdots \zeta_1)^2 = 1$$

ζ_i commutes with $\zeta_1 \cdots \zeta_{2g+1}^2 \cdots \zeta_1$



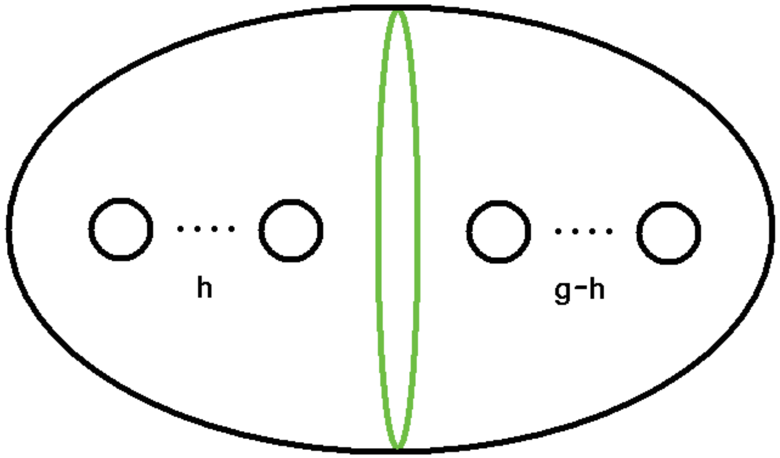
⋆ ζ_i conjugate each other

$$\phi(\zeta_i) = \frac{g+1}{2g+1} \quad \text{(for any } i\text{)}$$

$\star\ \psi_h = (\zeta_1 \cdots \zeta_{2h})^{4h+2} \in \Delta_g$

Dehn twist along a bounding simple closed curve

$$\phi(\psi_h) = -\frac{4}{2g+1} h(g-h)$$



BSCC map

$\boxed{g=1} \quad \Delta_1 \cong \mathcal{M}_1 = SL(2, \mathbf{Z})$

$\star$ Meyer, Kirby-Melvin, Sczech

$\cdots$ explicit formula of τ and ϕ

$\star$ Atiyah $\cdots$ geometric meanings of ϕ

Various inv. ass. to $SL(2, \mathbf{Z})$ coincide with ϕ

(cf. Rademacher's function, Hirzebruch signature defect, η invariant & its adiabatic limit, etc.)

$\boxed{g \geq 2}$ Geometric aspects of ϕ?

$\star$ periodic auto. (of finite order) $\Rightarrow$ η-invariant

(mapping torus)

$\star$ $\mathbf{Z}$-covering $\Rightarrow$ von Neumann ρ-invariant

(1^{st} MMM class & Rochlin inv)

$\star$ Torelli group $\Rightarrow$ Casson invariant

(Heegaard splitting)

Related works

$\star$ Kasagawa, Iida

$\cdots$ other construction of ϕ for $g = 2$

$\star$ Matsumoto, Endo

$\cdots$ the loc. sign. of hyp. Lefschetz fibrations

$\star$ Kuno, Sato

$\cdots$ Meyer's function in other settings

§ Eta-invariant

M : ori. closed Riem 3-mfd $\to \eta(M)$ is defined

Thm Atiyah-Patodi-Singer

W : a cpt ori Riem 4-mfd s.t. $\partial W = M$,

product near M

$$\eta(M) = \frac{1}{3}\int_W P_1 - \mathrm{Sign}\ W$$

P_1 : 1^{st} Pontrjagin form of the metric

Remark If W is closed $\Rightarrow \text{Sign } W = \frac{1}{3}\int_W P_1$

For $f \in \mathcal{M}_g$

$M_f = \Sigma_g \times \mathbf{R}/(x,t) \sim (f(x), t+1)$ mapping torus

Thm $f \in \Delta_g$ periodic $\Rightarrow \eta(M_f) = \phi(f)$

$$\begin{array}{cl} \Sigma_g \times S^1 & \\ \downarrow & \quad \text{finite Riem cov} \\ M_f & \end{array}$$

Cor $f \in \mathcal{M}_g$ periodic, $f \in \Delta_g \Rightarrow \eta(M_f) \in \frac{1}{2g+1}\mathbf{Z}$

Example there exists $f \in \mathcal{M}_3$ of order 3

s.t. its quotient orbifold $\approx S^2(3,3,3,3,3)$

Then direct computation shows

$$\eta(M_f) = -\frac{2}{3} \notin \frac{1}{7}\mathbf{Z}$$

$\Rightarrow f \notin \Delta_3$

§ **Relation to von Neumann rho-invariant**

Γ : a discrete group

M : an ori closed Riem 3-mfd

$\pi_1 M \to \Gamma$: a surjective homo

$\Rightarrow$ $\Gamma \to \hat{M} \to M$: Γ-covering

$\longrightarrow \eta^{(2)}(\hat{M})$ is defined von Neumann or

L^2 η-invariant

Def & Thm Cheeger-Gromov

$\underline{\eta^{(2)}(\hat{M}) - \eta(M)}$ is independent of a Riem metric

$\quad \| $

$\rho^{(2)}(\hat{M})$ von Neumann rho-invariant

Remark $\rho^{(2)}(\hat{M})$ is an extension of rho-invariant

η_γ : the η-invariant ass. to $\gamma : \pi_1 M \to U(n)$

$\Rightarrow \rho = \eta_\gamma - n\eta$ is independent of a Riem metric

For $f \in \Delta_g$

$\mathbf{Z} \to \hat{M}_f \to M_f$ $\mathbf{Z}$-covering associated to

$$\pi_1 M_f \to \pi_1 S^1$$

$\star$ ϕ is not multiplicative for coverings

Thm $\rho^{(2)}(\hat{M}_f) = \lim_{k\to\infty} \dfrac{\phi(f^k) - k\phi(f)}{k}$

Using the thm stated before and the approximation thm of the η-inv, due to Vaillant, Lück-Schick

Example $g = 1 \quad A \in SL(2, \mathbf{Z})$

(1) Elliptic case ($|\mathrm{tr}\ A| < 2$)

Let $A_n \in SL(2, \mathbf{Z})$ have the order n

$$A_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \ A_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ A_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\rho^{(2)}(\hat{M}_{A_n}) = \begin{cases} 2/3 & n = 3 \\ 1 & n = 4 \\ 4/3 & n = 6 \end{cases}$$

(2) Parabolic case ($|\mathrm{tr}\ A| = 2$) $A_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ $(b \in \mathbf{Z})$

$$\rho^{(2)}(\hat{M}_{A_b}) = -\mathrm{sgn}(b) = \begin{cases} -b/|b| & b \neq 0 \\ 0 & b = 0 \end{cases}$$

(3) Hyperbolic case ($|\mathrm{tr}\ A| > 2$)

$\rho^{(2)}(\hat{M}_A) = 0$ ($\phi(A^k) = k\phi(A)$ holds)

Cor If $f \in \mathcal{I}_g \cap \Delta_g \Rightarrow \rho^{(2)}(\hat{M}_f) = 0$

(ϕ is a homomorphism on $\mathcal{I}_g \cap \Delta_g$)

Remark If we restrict the above thm to the level 2 subgroup, we can obtain a relation among von Neumann rho-inv, 1^{st} MMM class and Rochlin inv in a framework of the bdd cohomology

$$\left(\begin{array}{l} f^* e_1 \text{``}=\text{''} \mu(M_f) - \rho^{(2)}(\hat{M}_f) \text{ in } H_b^2(S^1, \mathbf{Z}) \cong \mathbf{R}/\mathbf{Z} \\ f \in \mathcal{M}_g(2) = \ker\{\mathcal{M}_g \to \mathrm{Sp}(2g, \mathbf{Z}/2)\} \end{array} \right)$$

§ **Casson invariant** $g \geq 2$

λ : $\{M \mid \text{ori homology 3-sphere}\} \quad \longrightarrow \quad \mathbf{Z}$

$\lambda(M) \sim \#\{\pi_1 M \to SU(2) \text{ irr rep}\}/\text{conj}$

$\star$ Theory of characteristic classes of surface bundles we can consider the Casson inv of $\mathbf{Z}HS^3$ from the view point of $\mathcal{M}_g$ $\;$ Morita

$\mathcal{K}_g = \langle \text{BSCC map} \rangle \subset \mathcal{I}_g$

bounding simple closed curve

Fix a Heegaard splitting of S^3

$S^3 = H_g \cup_{\iota_g} -H_g \ (\iota_g \in \mathcal{M}_g)$

H_g : handle body of genus g

$$
\begin{array}{ccc}
\mathcal{K}_g \ni f & \longmapsto & M^f = H_g \cup_{\iota_g f} -H_g \\
\lambda^* \searrow & & \swarrow \mathbf{Z}HS^3 \\
& \mathbf{Z} \ni \lambda(M^f) &
\end{array}
$$

$\lambda^* \cdots$ sum of two homomorphisms

Morita's homo $d_0 : \mathcal{K}_g \to \mathbf{Q}$ core of Casson inv

Johnson's homo $\cdots$ Massey products

Thm $\phi = \dfrac{1}{3} d_0$ on $\Delta_g \cap \mathcal{K}_g$

Example $\psi_h \in \Delta_g \cap \mathcal{K}_g$: a BSCC map of genus h

$$\begin{aligned} d_0(\psi_h) &= 3\phi(\psi_h) \\ &= -\frac{12}{2g+1} h(g-h) \end{aligned}$$

1^{st} Mumford-Morita-Miller class $e_1 \in H^2(\mathcal{M}_g, \mathbf{Z})$

$E \xrightarrow{\pi} X$: oriented Σ_g bundle

$T\pi = \{v \in TE \mid \pi_* v = 0\}$: tangent bundle along the fiber

$e = \mathrm{Euler}(T\pi) \in H^2(E, \mathbf{Z})$

$\pi_! : H^4(E, \mathbf{Z}) \to H^2(X, \mathbf{Z})$ Gysin homomorphism

$\Rightarrow \; e_1 = \pi_!(e^2) \in H^2(X, \mathbf{Z})$ the 1^{st} MMM class

$$H^2(\mathrm{BDiff}_+\Sigma_g, \mathbf{Z}) = H^2(K(\mathcal{M}_g, 1), \mathbf{Z}) = H^2(\mathcal{M}_g, \mathbf{Z})$$

($\mathrm{Diff}_0\Sigma_g$ contractible for $g \geq 2$ Earle-Eells)

$\Rightarrow\ e_1 \in H^2(\mathcal{M}_g, \mathbf{Z})$

$\star$ There exist canonical 2-cocycles representing $e_1/\mathbf{Q}$

- -3τ : signature cocycle

- c : intersection cocycle

  (fix a crossed homomorphism of $\mathcal{M}_g$)

there exists uniquely defined mapping $d : \mathcal{M}_g \to \mathbf{Q}$

s.t. $\delta d = c + 3\tau$

Fact Morita

$$d_0 = d|_{\mathcal{K}_g} \begin{cases} \text{does not depend on the choice of} \\ \qquad\qquad\qquad \text{crossed homomorphisms} \\ \text{is a generator of } H^1(\mathcal{K}_g, \mathbf{Z})^{\mathcal{M}_g} \end{cases}$$

$d_0 : \mathcal{K}_g \to \mathbf{Q}$ Morita's homomorphism

Appendix Bounded cohomology

G : a discrete group, $A = \mathbf{R}, \mathbf{Z}$

$C_b^*(G) = \{c : G \times \cdots \times G \to A \mid \text{the range is bdd}\}$

$\delta : C_b^p(G) \to C_b^{p+1}(G)$

$$\begin{aligned}\delta c(g_1, \ldots, g_{p+1}) = c(g_2, \ldots, g_{p+1}) - c(g_1 g_2, g_3, \ldots, g_{p+1}) \\ \cdots + (-1)^p c(g_1, \ldots, g_p g_{p+1}) \\ + (-1)^{p+1} c(g_1, \ldots, g_p)\end{aligned}$$

$H_b^*(G, A) = H^*(C_b^*(G), \delta)$ bounded cohomology

We want to consider e_1 for a surface bdl over S^1. However, for a holonomy homo $f : \pi_1 S^1 \to \mathcal{M}_g$, $f^* e_1 = 0$, because $H^2(S^1, \mathbf{Z}) = 0$.

Fact

(1) e_1 is a bounded cohomology class

(2) $H_b^2(S^1, \mathbf{Z}) \cong H_b^2(\mathbf{Z}, \mathbf{Z}) \cong \mathbf{R}/\mathbf{Z}$ Ghys

$\Rightarrow$ $f^* e_1$ might be <u>nontrivial</u> as a bdd class

Thm

If $\mathrm{Im}\{f : \pi_1 S^1 \to \mathcal{M}_g\} \subset \mathcal{M}_g(2)$

|| level 2 subgroup

$\ker\{\mathcal{M}_g \to \mathrm{Sp}(2g, \mathbf{Z}/2)\}$

$\Rightarrow \ f^* e_1 \text{``}=\text{''} \mu(M_f) - \rho^{(2)}(\hat{M}_f) \mod \mathbf{Z}$

Remark Kitano

If $\mathrm{Im} f \subset \mathcal{I}_g \ \Rightarrow \ f^* e_1$ is given by the Rochlin inv

Combining Thm and the result of Miller-Lee

Cor f^*e_1 is a spectral invariant i.e.
$$f^*e_1 \text{``}=\text{''} \eta^{(2)}(\hat{M}_f) + \hbar + \eta_{\mathcal{D}}(M_f) \mod \mathbf{Z}$$

$\mathcal{D}$: Dirac operator acting on the spinor fields

$\hbar$: dim of the space of harmonic spinors