On Charts with Two Crossings

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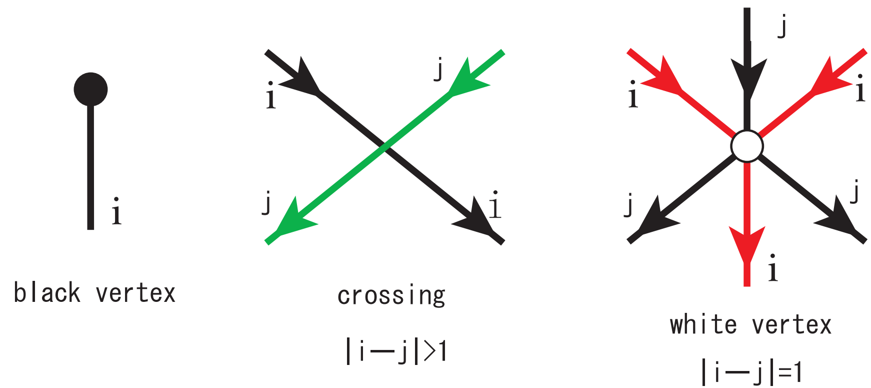
An n-chart Γ is an oriented labeled graph on the disk s.t. each label of edges is 1,2, $\cdots$, or $n-1$, and each vertex is one of the followings:



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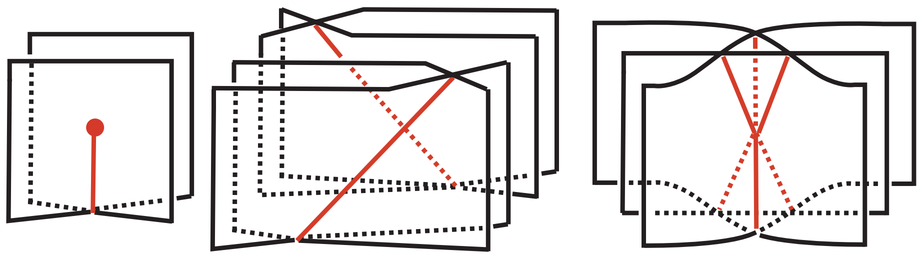
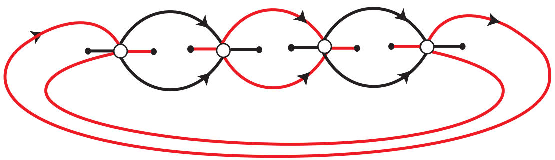
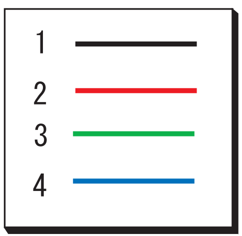


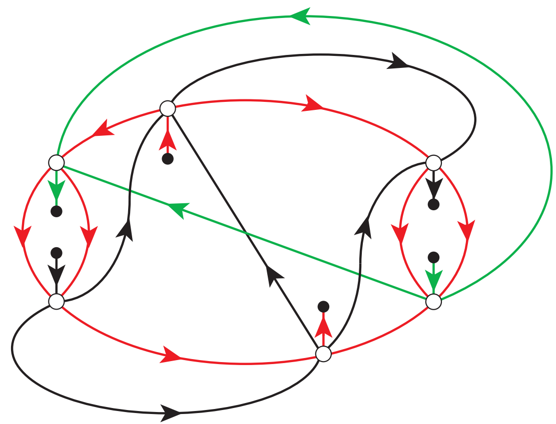
Examples:

3-chart

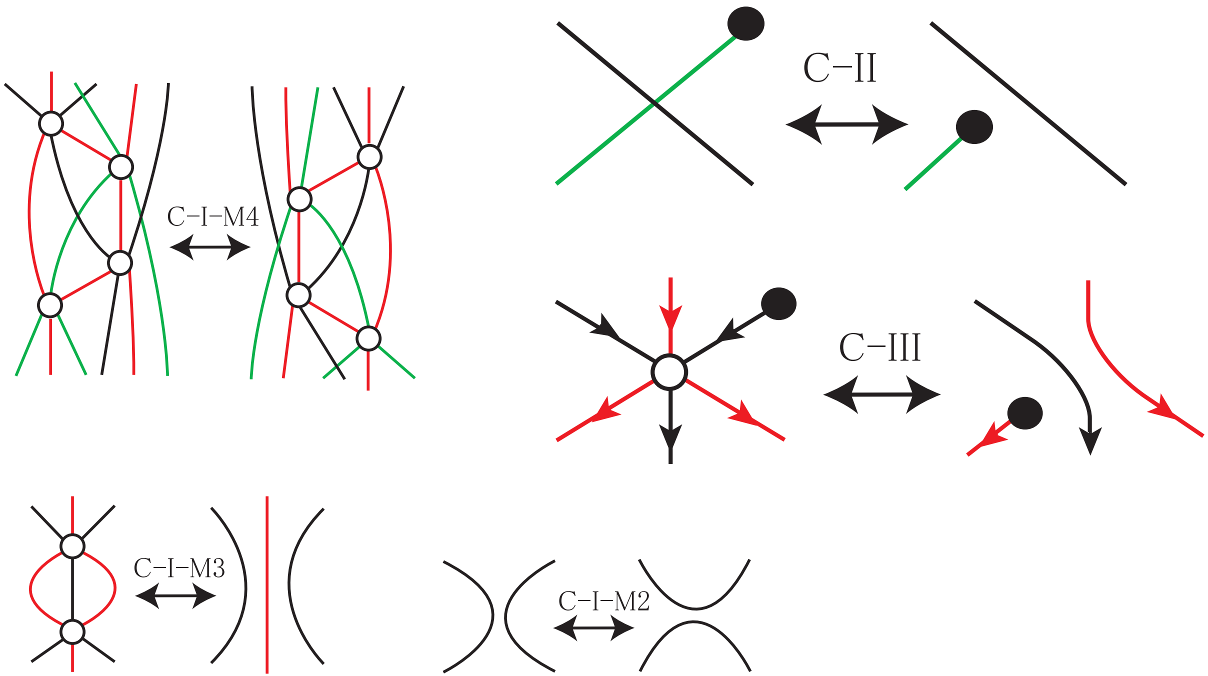




4-chart

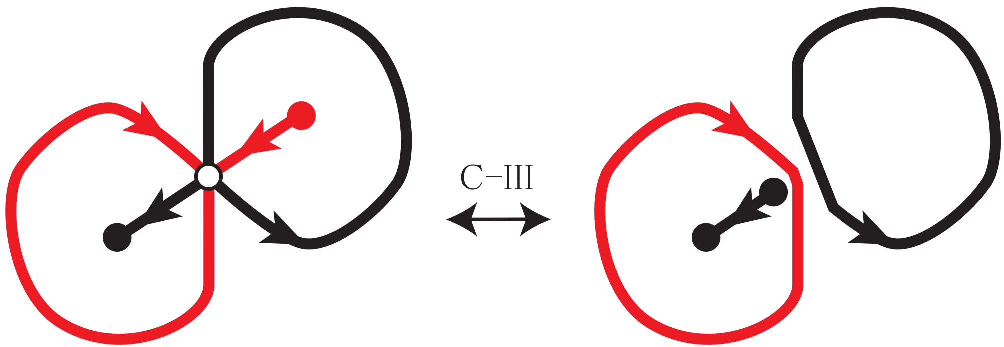


C-moves:



A chart Γ' is said to be C-move equivalent to a chart Γ if Γ' is obtained from Γ by a finite sequence of C-moves.

A ribbon chart is C-move equivalent to a chart without white vertices.



[**Kamada**] Any 3-chart is a ribbon chart.

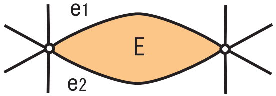
[**Nagase and Hirota**] Any 4-chart with at most one crossing is a ribbon chart.

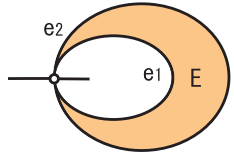
[**Nagase and Shima**]

(1) Any 2-minimal 4-chart with exactly two crossings contains at least 8 black vertices.

(2) If a 4-chart contains at most two crossing, and if it represents one sphere, then it is a ribbon chart.

(3) Any chart with at most one crossing is a ribbon chart.





free edge

Let Γ be a chart.

$w(\Gamma)$ = the number of white vertices,

$f(\Gamma)$ = the number of free edges,

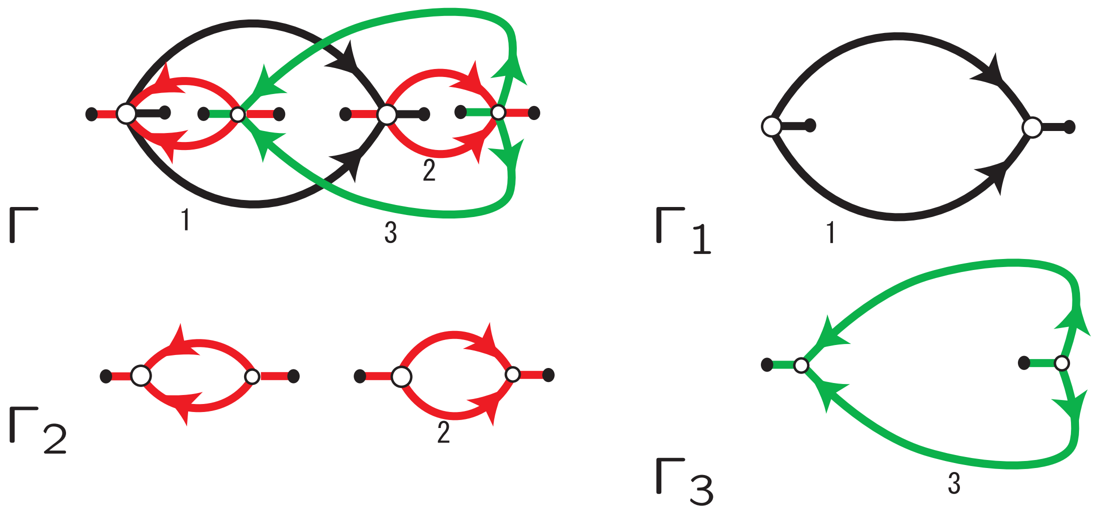
$b(\Gamma)$ = the number of bigons.

The extended complexity of Γ is $(w(\Gamma), -f(\Gamma), -b(\Gamma))$.

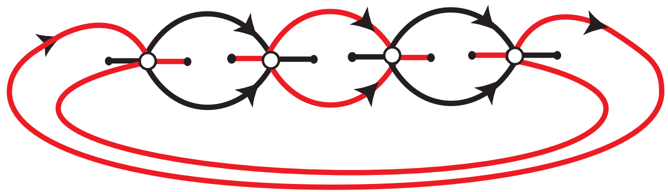
Let Γ be a chart with at most k crossings.

A k-minimal chart Γ has the minimal complexity among the charts with at most k crossings C-move equivalent to Γ.

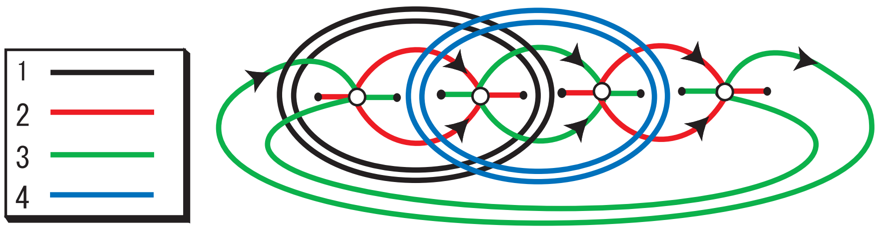
For each label m, we denote by Γ_m the subgraph of a chart Γ consisting of edges of label m and their vertices.



A chart Γ is called a generalized n-chart if $\exists\, p$ and q s.t. $n = q - p$, $w(\Gamma_{p+1}) \neq 0$, $w(\Gamma_{q-1}) \neq 0$, and $w(\Gamma_i) = 0$ except for $p < i < q$ where $w(X) =$ the number of white vertices in X.



3-chart



generalized 3-chart (5-chart)

Main Theorem. Let $n > 4$.
(1) If a 2-minimal generalized n-chart with exactly two crossings contains at least $4n-10$ black vertices.
(2) If an n-chart contains at most two crossings, and if it represents a disjoint union of spheres, then it is a ribbon chart.

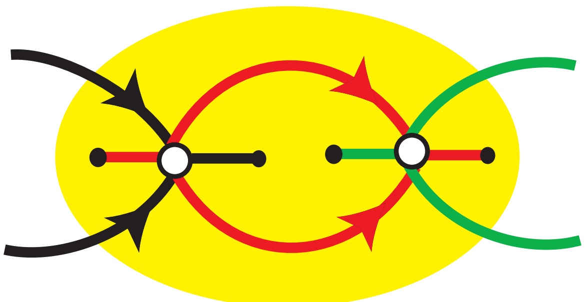
Remark. If an n-chart represents one sphere, then it contains exactly $2n-2$ black vertices.

Let Γ be a chart and D a disk.

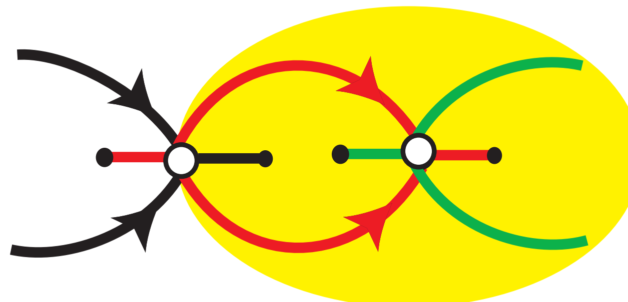
The pair $(D \cap \Gamma, D)$ is called a tangle if

(1) ∂D does not contain any vertices of Γ,

(2) ∂D transversely intersects edges of Γ.

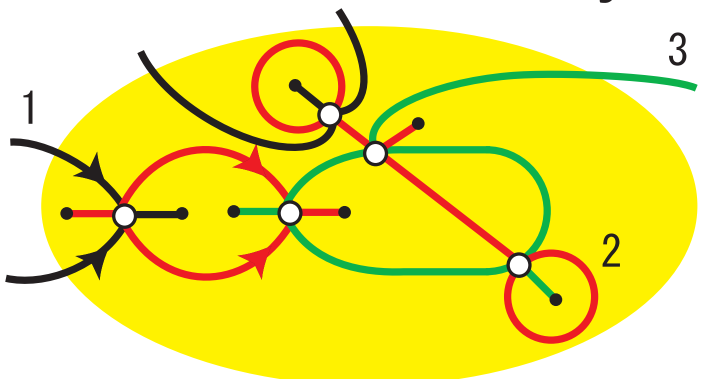


a tangle



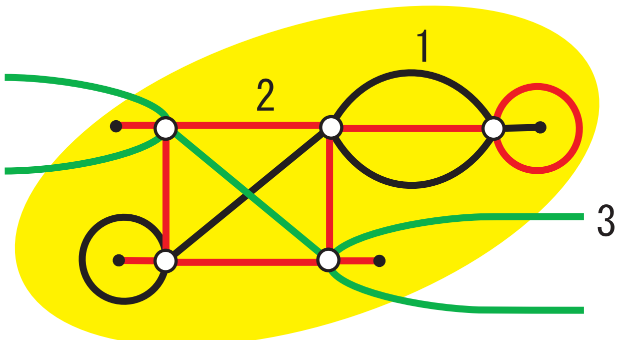
not a tangle

A tangle $(D \cap \Gamma, D)$ is called an NR-tangle (new reducible tangle) of label m if (1) $\partial D \cap \Gamma$ is contained in Γ_m except at most one point, (2) D contains at least one white vertex, but D does not contain any crossings.



NR-tangle of label 1

A tangle $(D \cap \Gamma, D)$ is called an NS-tangle of label m if (1) $\partial D \cap \Gamma \subset \Gamma_m$,
(2) D contains at least one white vertex, and D contains at most one crossing.

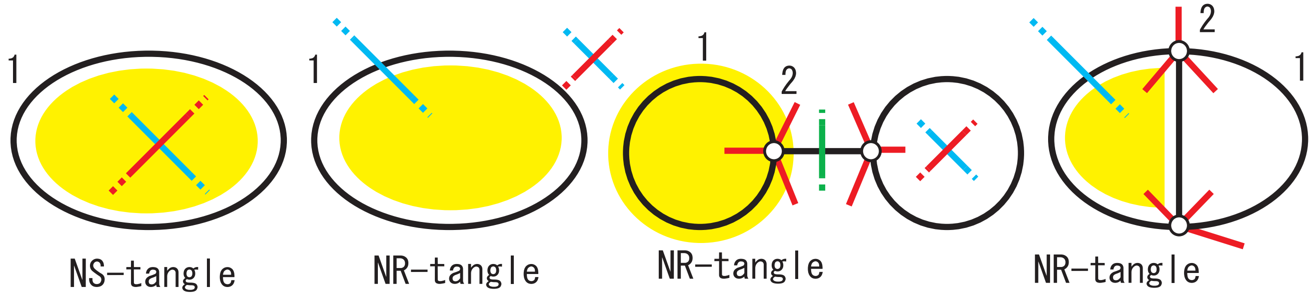


an NS-tangle of label 3

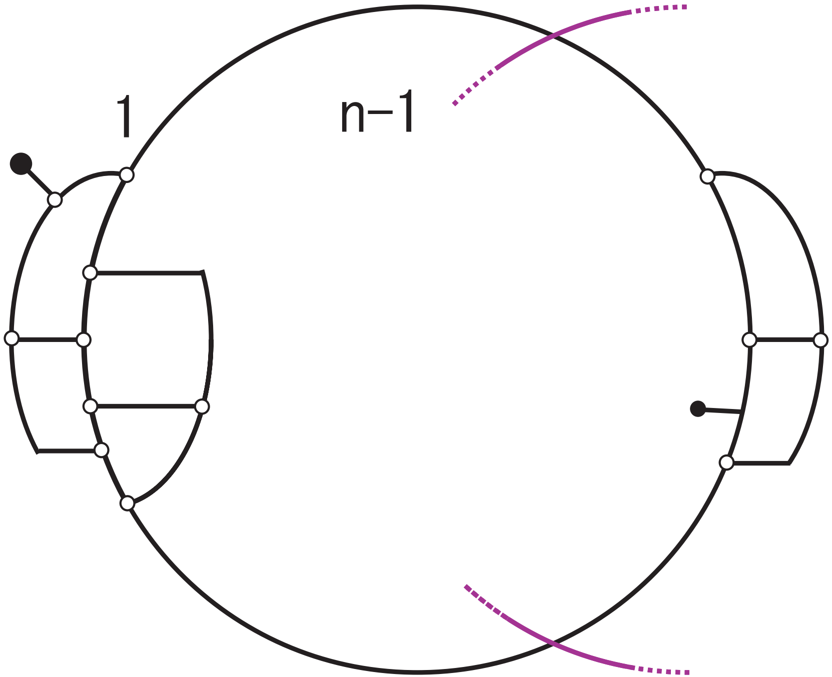
Lemma. There exists neither an NR-tangle nor an NS-tangle in any k-minimal chart.

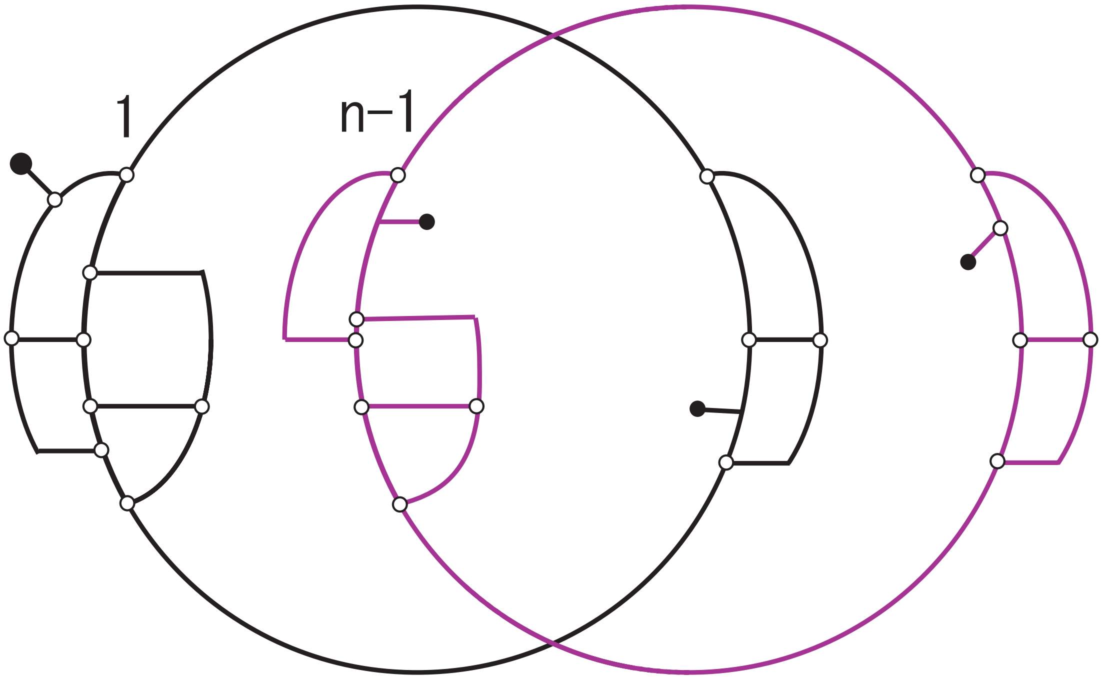
Outline of Proof of Main Theorem.

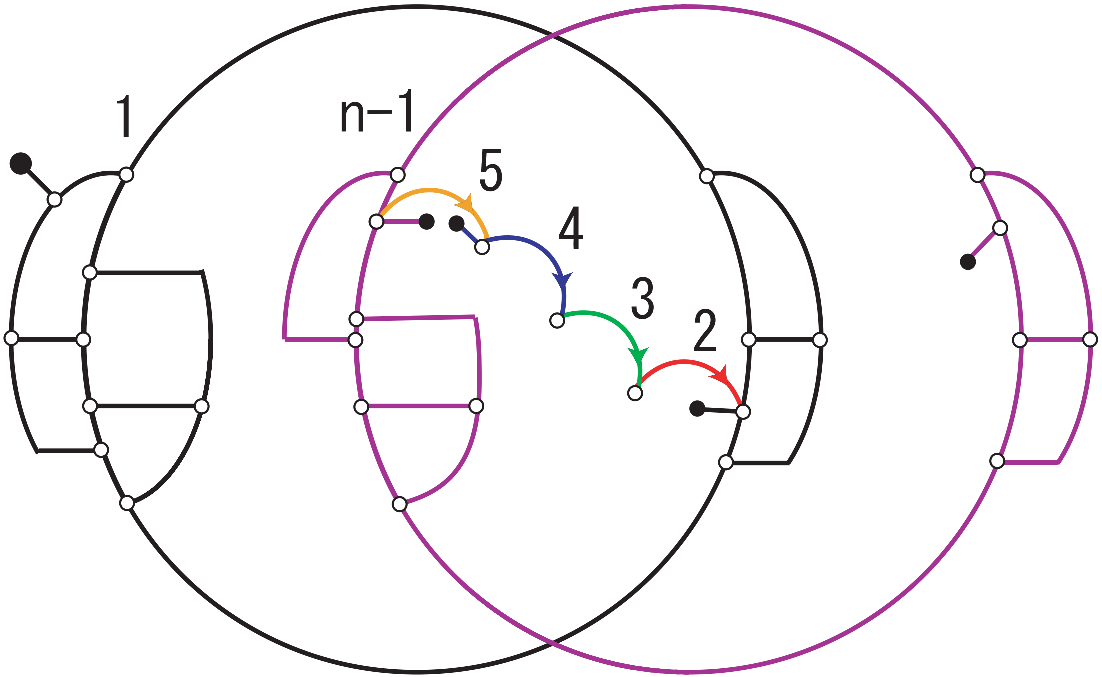
Γ : a 2-minimal n-chart with two crossings with $w(\Gamma_1) \neq 0$ and $w(\Gamma_{n-1}) \neq 0$.

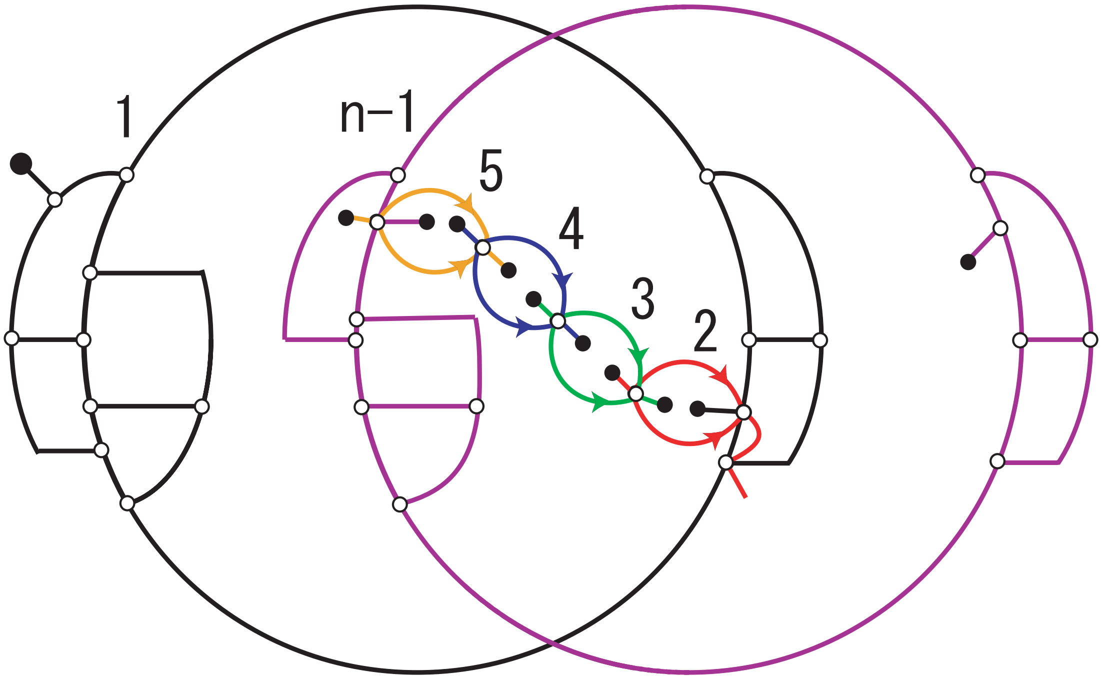


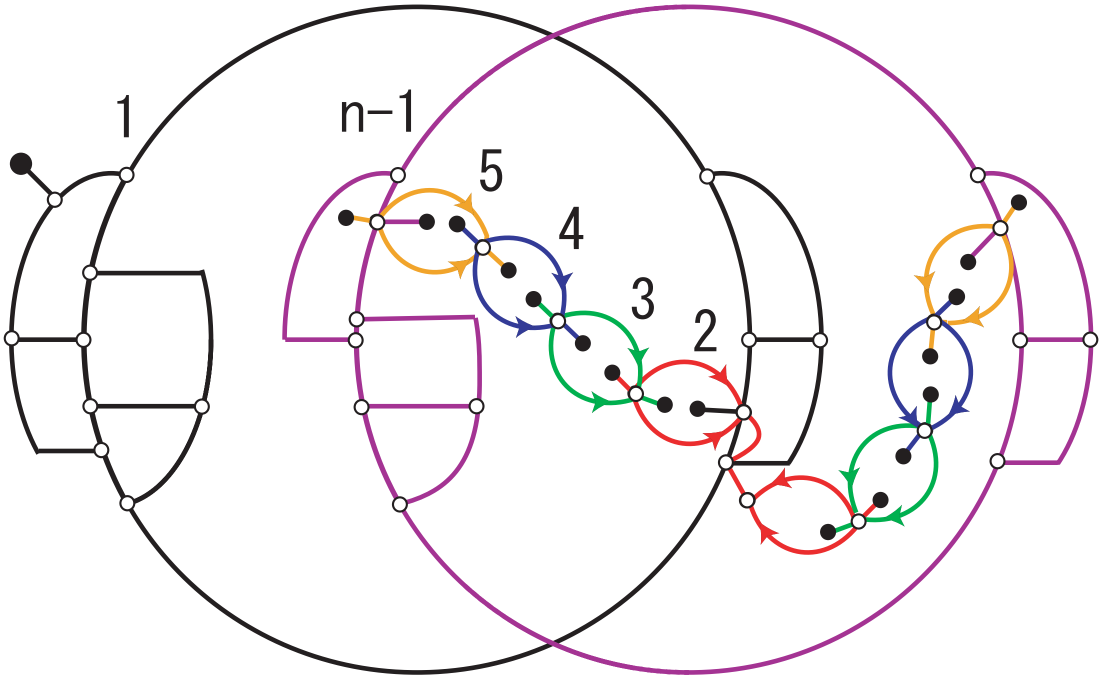
Claim: There exist simple closed curves C and C' in Γ_1 and Γ_{n-1} respectively s.t. $C \cap C'$ contains two crossings.


1
n−1


1
n−1


1
n−1
5
4
3
2


1
n−1
5
4
3
2



$4(n-4)+6=4n-10$ black vertices.

Lemma. Let Γ be a k-minimal chart and G a 'small' component of Γ_m. If G is contained in a disk such that the disk does not contain any crossings, then G contains at least two black vertices.

Roughly speaking, a small component means an innermost component in Γ_m.