The Fourth East Asian School of Knots

and Related Topics

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An infinite family of exotic 4-manifolds and Rasmussen invariants of knots

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§1. Casson handles

§2. Known results

§3. Rasmussen's s-invariant

§4. Results

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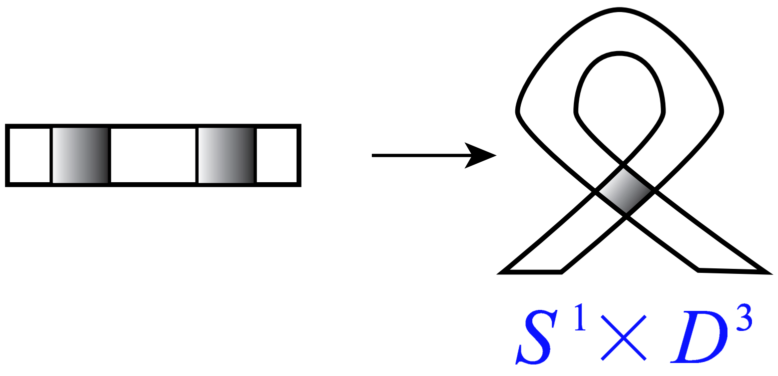
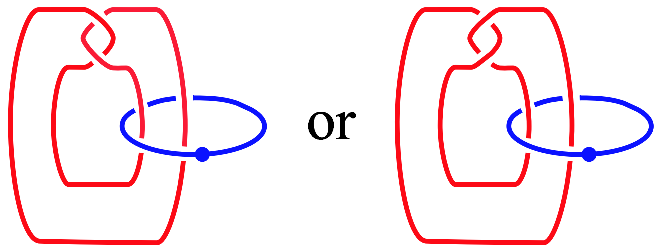
§1. Casson handles

A kinky handle:

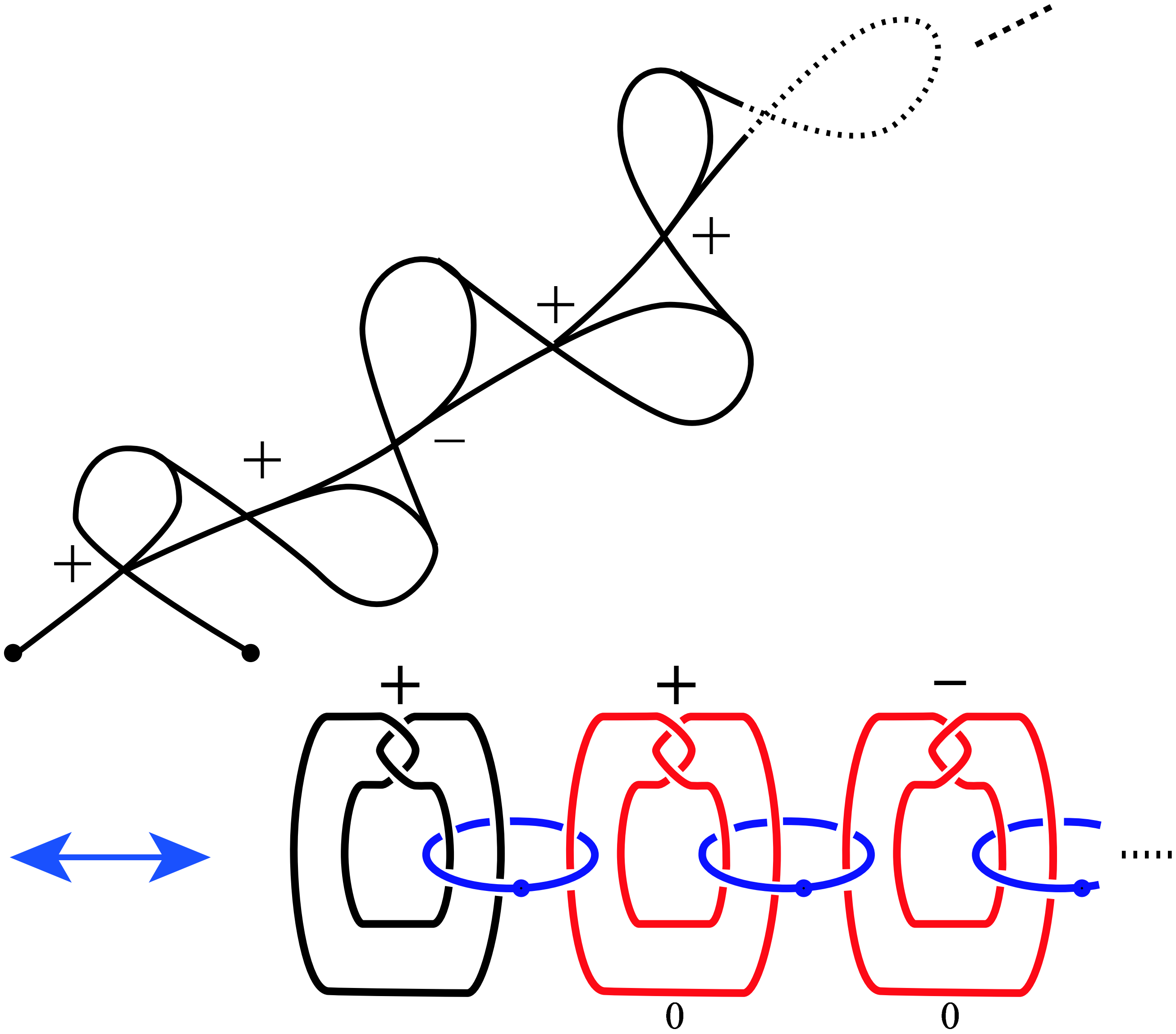


The attaching region :

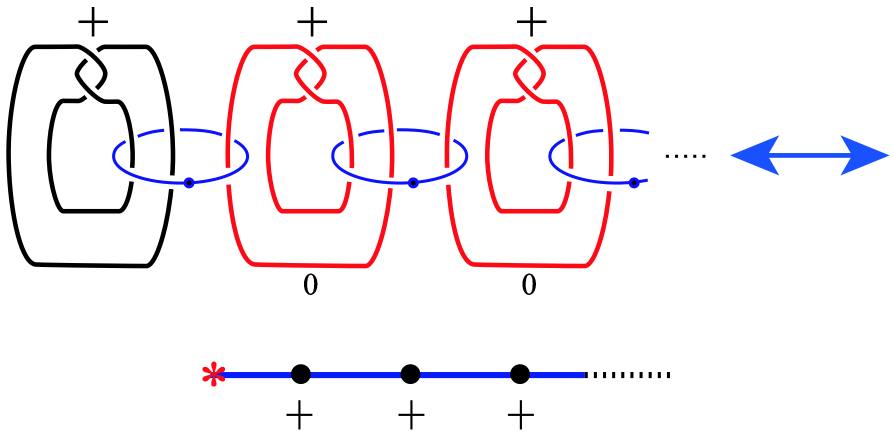


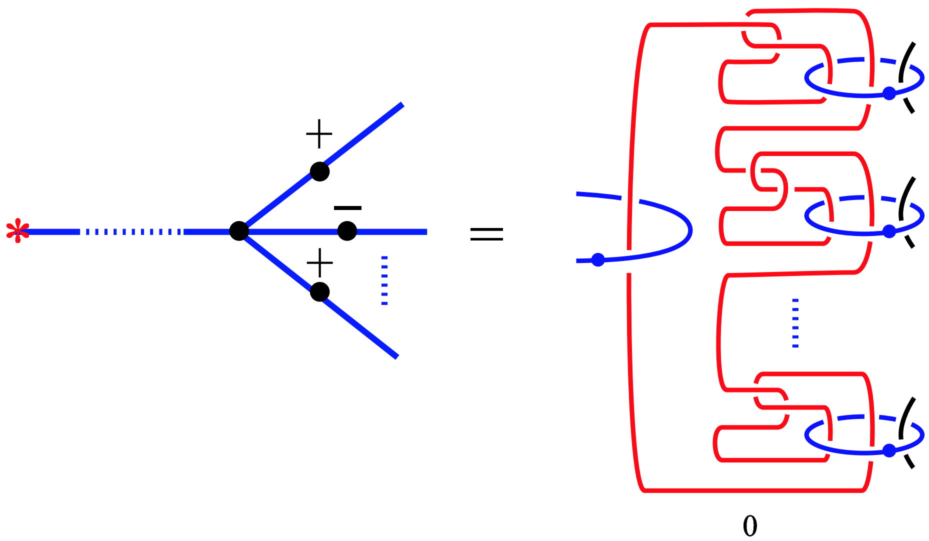
A kinky handle is a 2-handle modulo a finite number ($\neq 0$) of self-plumbings.

A Casson handle:



A Casson handle $\Longleftrightarrow$ infinite-based signed tree





§2. Known results

Theorem. (M. Freedman (1983)) Any Casson handle is homeomorphic to the standard open 2-handle.

Theorem. (R. Gompf (1984)) There exist countably many Casson handles.

Theorem. (R. Gompf (1989)) There exist uncountably many Casson handles.

A Casson handle is exotic if the attaching circle does not bound a smooth 2-disc in the Casson handle.

Problem.	Is any Casson handle exotic?

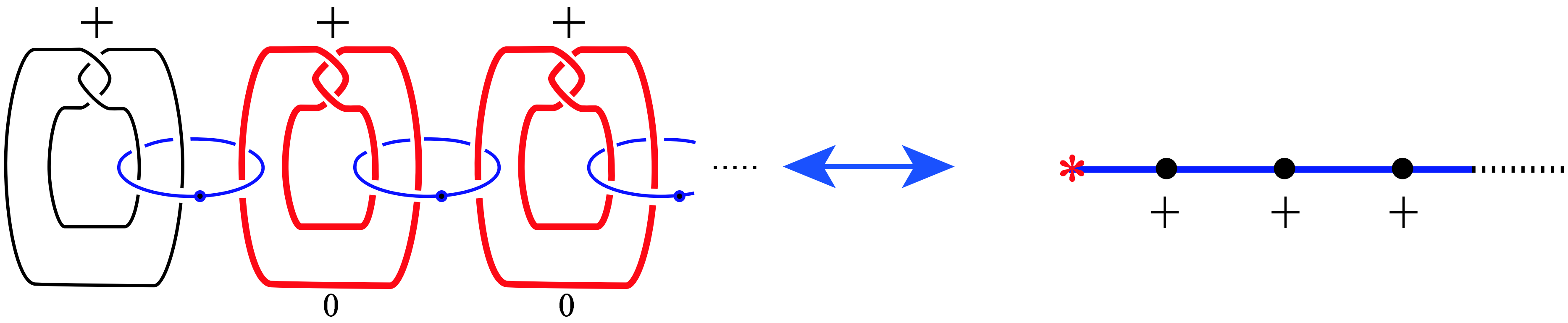
Fact. T, T': infinite-based signed trees.

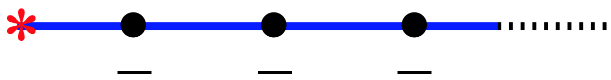
$$T \subset T' \Rightarrow CH_{T'} \subset CH_T.$$

Thus if CH_T is exotic, then $CH_{T'}$ is also exotic.

Ž. Bižaca showed an explicit example of

an exotic Casson handle (1995).

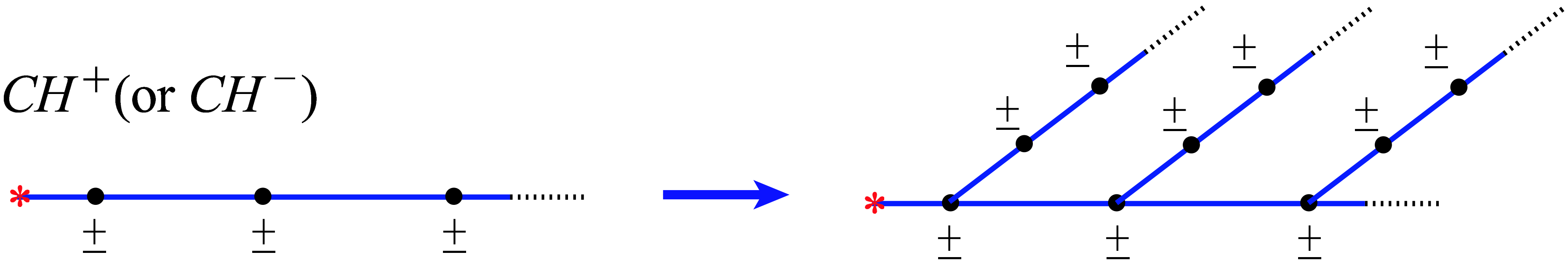


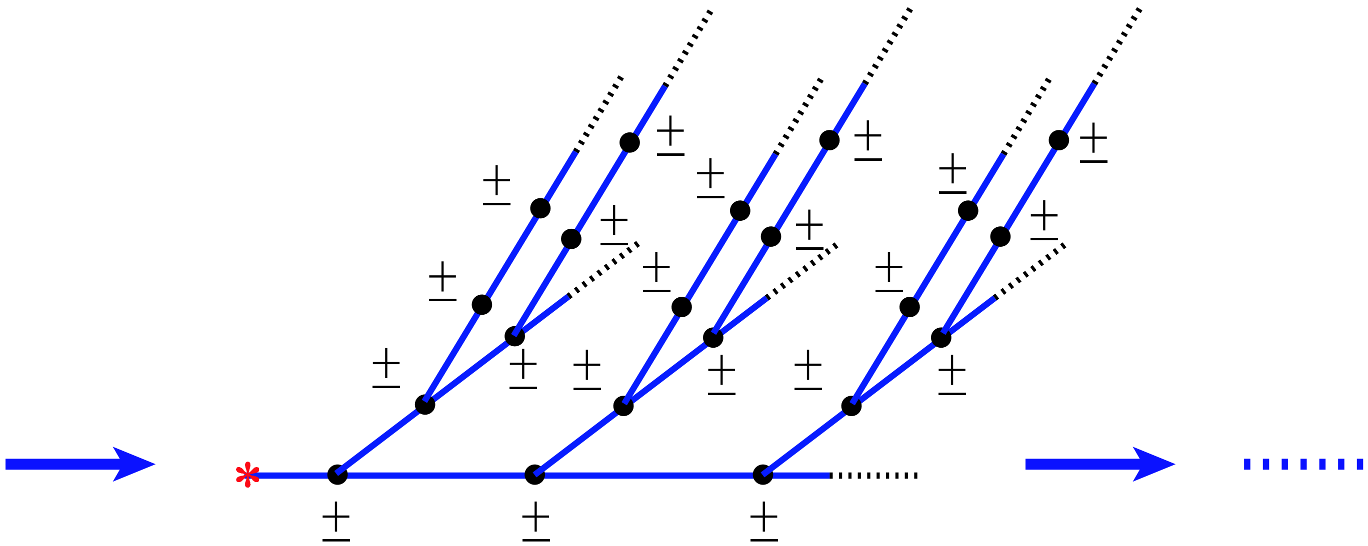


: periodic Casson handles.

[Casson handle of boundary type]

CH^+ (or CH^-)





Remark. Any Casson handle of boundary type is exotic.

Theorem. (T. Kato (2006)) There exists a Casson handle which is not boundary type.

§3. Rasmussen's s-invariant

J. Rasmussen, Khovanov homology and the slice genus, math.GT/0402131, 2004
(to appear in Inventiones Mathematicae.)

Lee's variant of Khovanov homology $\Longrightarrow$
a concordance invariant s of a knot (combinatorially).

[Knot concordance group]

A knot K is slice $\Longleftrightarrow$ K bounds a smooth disc in B^4.

Knots K_1 and K_2 are concordant $\Longleftrightarrow K_1 \sharp - K_2$ is slice.

The set $\{concordance\ classes\}$ forms a abelian group under $\sharp$ (the knot concordance group).

[Slice genus]

F: a smooth conn. ori. surface properly embedded in B^4 with boundary K.

$g_s(K)$:=min{the genus of F} (the slice genus of K).

Theorem (J. Rasmussen)

Let K be a knot in S^3. Then

(1) s induces a homomorphism from the knot concordance group to $\mathbb{Z}$;

(2) $|s(K)| \leq 2g_s(K)$;

(3) If K is alternating, then $s(K) = \sigma(K)$, where $\sigma(K)$ is the classical knot signature of K;

(4) $s(T_{p,q}) = u(T_{p,q}) = (p-1)(q-1)/2$ (Milnor conjecture).

§4. Results

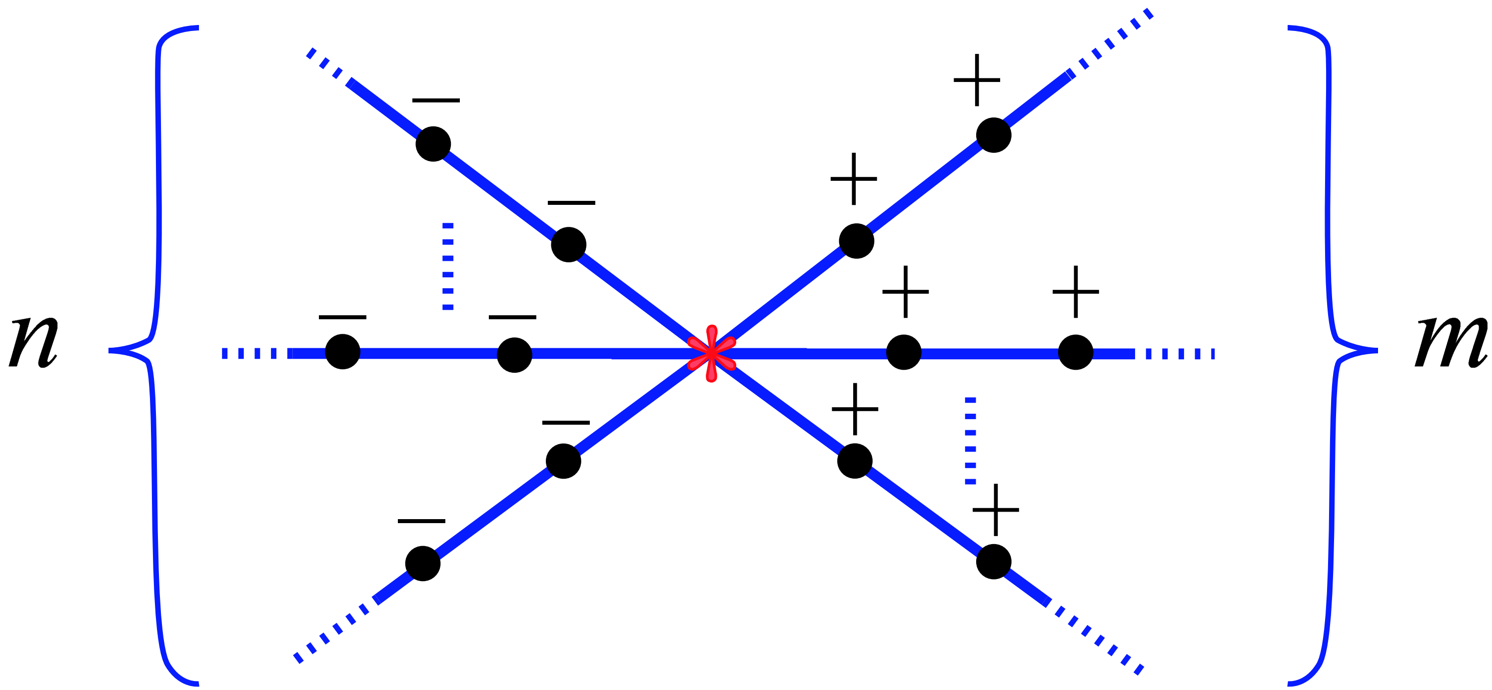
CH: a Casson handle.

F: a smooth conn. ori. surface properly embedded in CH with boundary the attaching circle.

$g_s(CH)$:=min{the genus of F} (the slice genus of CH).

Problem. Does there exists a Casson handle CH satisfying $g_s(CH) > n$ for any positive integer n?

$$CH_{m,n} =$$



By using Rasmussen invariant, we show the following:

Theorem 1. Let m_i and n_i be non-negative integers with $m_i + n_i \neq 0$ $(i = 1, 2)$. If $m_1 + n_1 < |m_2 - n_2|$, then $|m_1 - n_1| \leq g_s(CH_{m_1,n_1}) < g_s(CH_{m_2,n_2}) \leq m_2 + n_2$.

Remark. Any $CH_{m,n}$ is exotic.

Proof of Theorem 1. It suffice to show that

$|m-n| \leq g_s(CH_{m,n}) \leq m+n$ if $m+n \neq 0$.

Claim 1. $g_s(CH_{m,0}) \leq m+n$.

Claim 2.

T_m: the connected sum of $c_i(>0)$-fold untwisted positive doubles $(1 \leq i \leq m)$ of the positive trefoil knot.

T_n: the connected sum of $c_i(>0)$-fold untwisted negative doubles $(m+1 \leq i \leq n)$ of the negative trefoil knot.

Let $T_{m,n} = T_m \sharp T_n$.

$|m-n| \leq g_s(T_{m,n}) \Rightarrow |m-n| \leq g_s(CH_{m,n})$.

Claim 3. $|m - n| \leq g_s(T_{m,n})$.

By using Rasmussen invariant, we can prove this claim.

Corollary 2. If $m_1 + n_1 < |m_2 - n_2|$, then

CH_{m_1,n_1} does not embed in CH_{m_2,n_2}.

Corollary 3. For any positive integer n,

there exist countably many Casson handles $\{CH_i\}_{i=0}^{\infty}$

such that $g_s(CH_i) \geq n$.

[Kinkiness of a knot]

K: a knot in S^3.

D: a normally immersed disc in B^4 which span K.

$k_\pm$ =min$\sharp\{$ positive (or negative) kinks in $D\}$

$\Longleftrightarrow$ the kinkiness of K.

[Kinkiness of a smooth 4-manifold]

V: a smooth 4-manifold.

C: a smoothly embedded circle in ∂V

with C null-homotopic in V.

(e.g. $(V, C) =$ a Casson handle.)

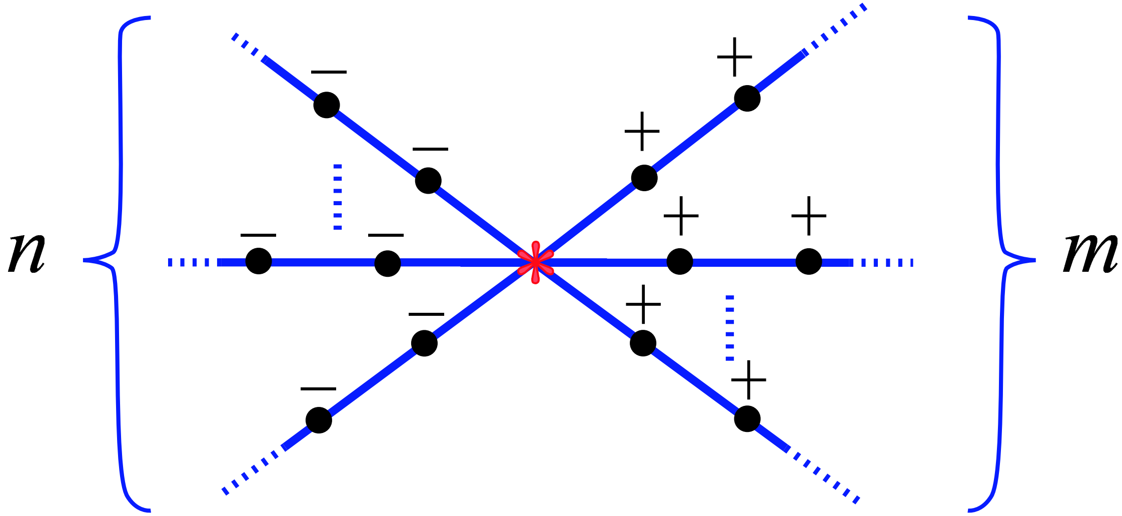
D: a normally immersed disc in V which span C.

$k_\pm = k_\pm(V, C) =$min$\sharp\{$ positive (or negative) kinks in $D\}$

$\Longleftrightarrow$ the kinkiness of (V, C).

Theorem 4. For any non-negative integers m and n with $m + n \neq 0$, we have $k(CH_{m,n}) = (m, n)$.

$$CH_{m,n} =$$



Proof of Theorem 4.

Claim 1. $k_+(CH_{m,0}) = k_+(CH_{m,n})$; $k_-(CH_{0,n}) = k_-(CH_{m,n})$.

Claim 2. $k_+(CH_{m,0}) \leq m$; $k_-(CH_{0,n}) \leq n$.

Claim 3. T_m: the connected sum of $c_i (> 0)$-fold untwisted positive doubles $(1 \leq i \leq m)$ of the positive trefoil knot.

$m \leq k_+(T_m)$; $n \leq k_-(-T_n) \Rightarrow$

$m \leq k_+(CH_{m,0})$; $n \leq k_-(CH_{0,n})$.

Claim 4. $m \leq k_+(T_m)$; $n \leq k_-(-T_n)$.

By using C. Bohr's inequality (2002) and Rasmussen invariant, we can show this claim.

§5. Other research

By using Rasmussen invariant, we show the following:

Theorem 5. Every non-compact, connected, oriented, smooth 4-submanifold of $\mathbb{R}^4$ admits at least two smooth structures.

By using a consequence of gauge theory (Donaldson's Theorem), we show the following:

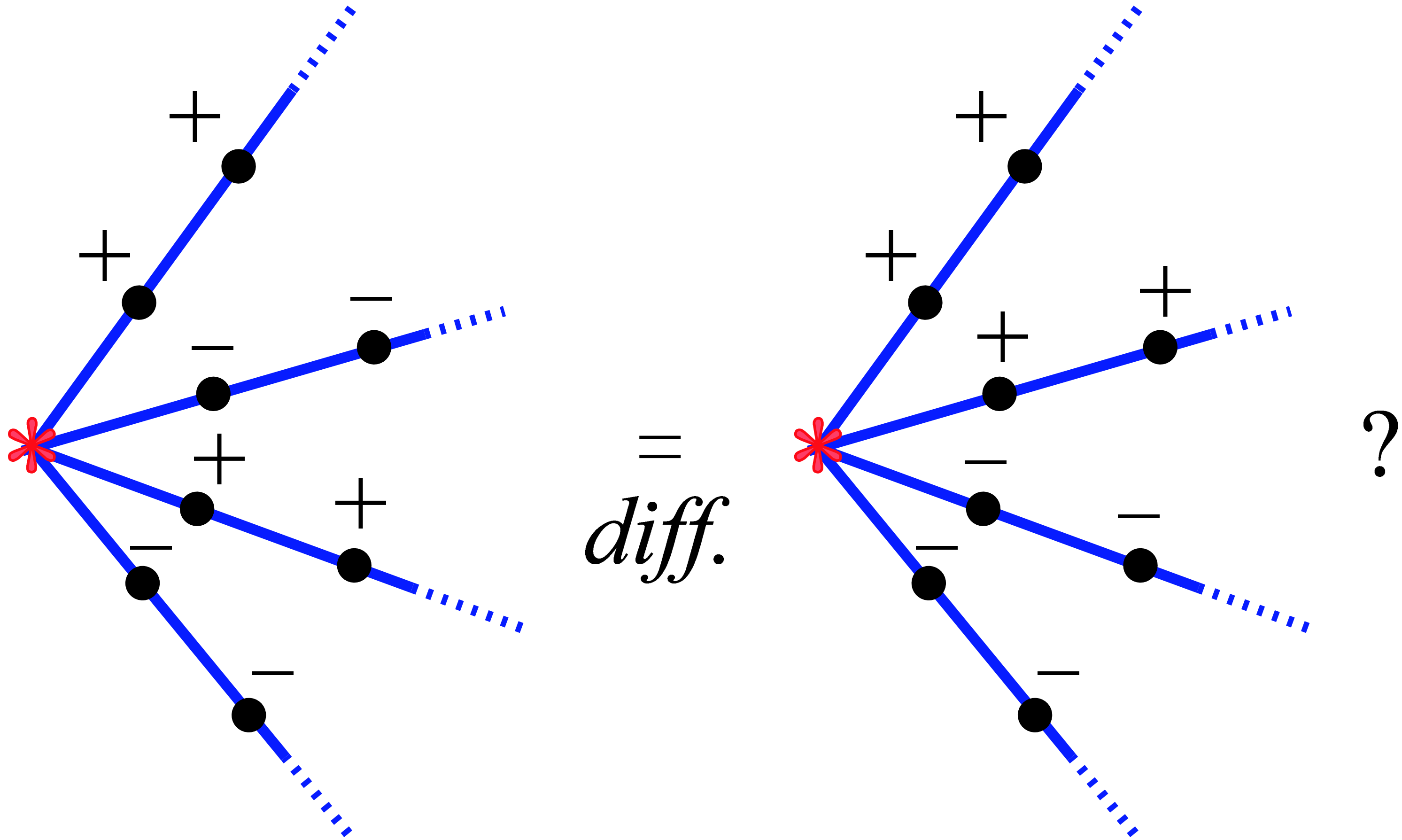
Theorem 6. Every non-compact, connected, oriented, smooth 4-submanifold of $\sharp_{i=1}^{\infty}\mathbb{CP}^2$ admits at least two smooth structures.

Corollary 7. For any positive integer n, every non-compact, connected, oriented, smooth 4-submanifold of $\sharp_{i=1}^{n}\mathbb{CP}^2$ admits at least two smooth structures.

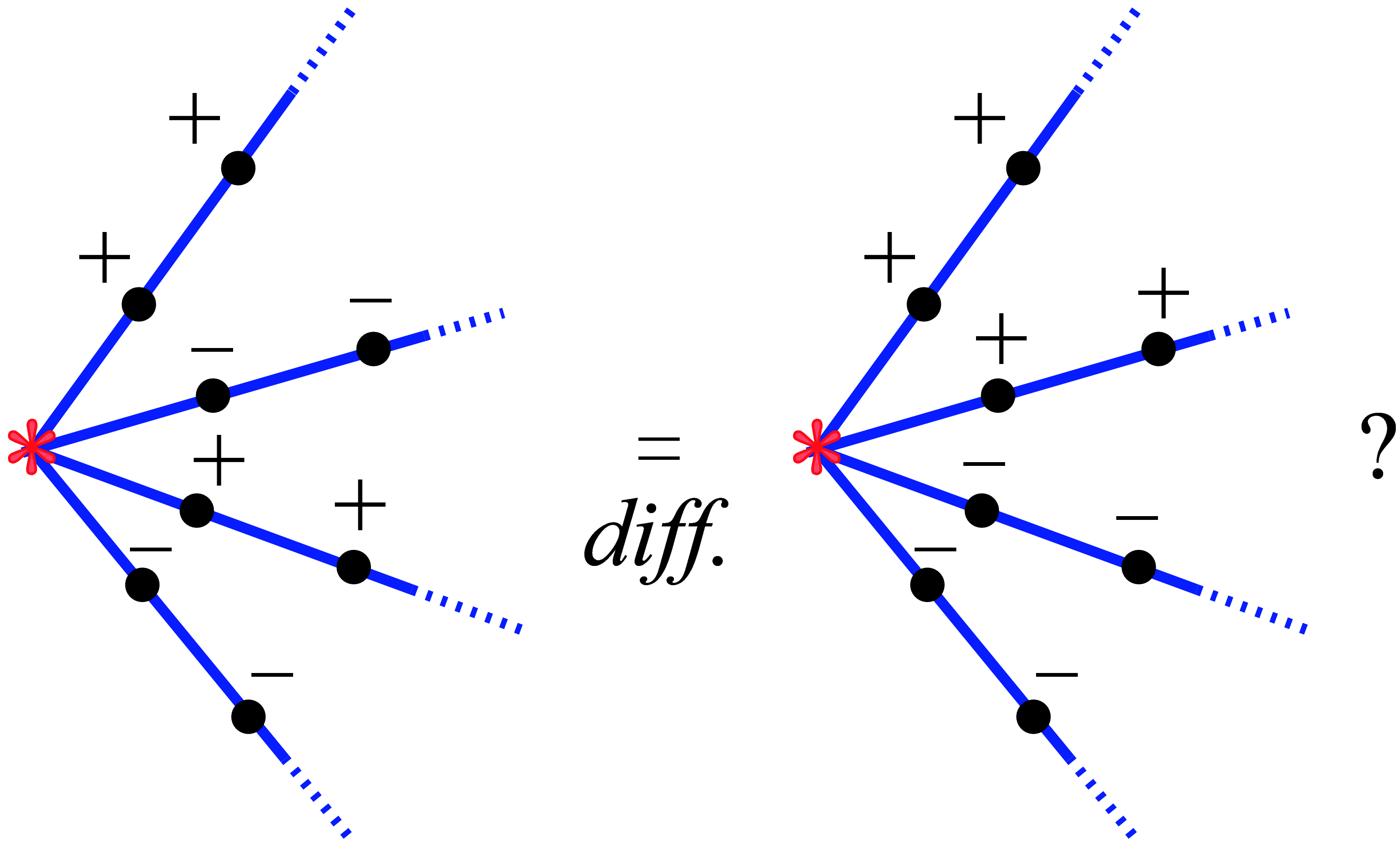
Problem (A. Kawauchi (1984)).

Does $\sharp_{i=1}^{\infty}\mathbb{S}^2 \times \mathbb{S}^2$ have at least two smooth structures?

§6. Problem



§6. Problem



Thank you very much.