An integral region choice problem on knot projection

Masaaki Suzuki

Akita University

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A region is an area surrounded by arcs in a knot diagram.

**Definition.**

A region crossing change at a region $R$ is the crossing change at all the crossings on $\partial R$.

**Example**

![Diagram showing a region crossing change](image)
Theorem (Shimizu).

A region crossing change on a knot diagram is an unknotting operation.

Example
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A region crossing change on a knot diagram is an unknotting operation.

Example

\[\begin{array}{c}
\includegraphics[width=0.3\textwidth]{example1.png}
\quad \Rightarrow
\quad \includegraphics[width=0.3\textwidth]{example2.png}
\end{array}\]

\[\begin{array}{c}
\includegraphics[width=0.3\textwidth]{example3.png}
\quad \Rightarrow
\quad \includegraphics[width=0.3\textwidth]{example4.png}
\end{array}\]
Another interpretation

Each crossing has been equipped with a score 0 or 1 modulo 2. If one chooses a region \( R \), then the scores of all the crossings which lie on \( \partial R \) are increased by 1 modulo 2. We can choose some regions so that the scores of all the crossings become 0.

Example
Integral extension

Each crossing has been equipped with an *integral score*. If one assigns an integer $n$ to a region $R$, then the scores of all the crossings which lie on $\partial R$ are increased by $n$.

**Problem (region choice problem).**

Can we assign integers to some regions so that the scores of all the crossings become 0?

**Example**

![Diagram showing a knot projection with integral scores before and after an assignment of integers to regions.](image-url)
Two rules: *single counting rule* and *double counting rule*

If one assigns an integer $n$ to a region $R$,

**single counting rule:**
scores of the crossings which lie on $\partial R$ are increased by $n$.

**double counting rule:**
scores of the crossings which $R$ touches *once* are increased by $n$,
scores of the crossings which $R$ touches *twice* are increased by $2n$.

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![Diagram](image)
Problem (region choice problem).
Can we assign integers to some regions so that the scores of all the crossings become 0 in the single and double counting rules?

Theorem (Ahara-S.).
The region choice problems of the single and double counting rule are solvable.

We can assign integers to some regions so that the score is increased by 1 at an arbitrary crossing without changing the scores of any other crossings.
add-1 operation
the score of the crossing $v$ is increased by 1 without changing the scores of any other crossings.
Problem (region choice problem).
Can we assign integers to some regions so that the scores of all the crossings become 0 in the single and double counting rules?

Theorem (Ahara-S.).
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We can assign integers to some regions so that the score is increased by 1 at an arbitrary crossing without changing the scores of any other crossings.
Definition (region choice matrix of double counting rule).

$\bar{D}$ : a knot projection

$R = \{r_1, r_2, \ldots, r_{n+2}\}$ : the set of regions

$V = \{v_1, v_2, \ldots, r_n\}$ : the set of crossings

\[
A(\bar{D}) = (a_{ij}) \in M_{n,n+2}(\mathbb{Z})
\]

where

\[
a_{ij} = \begin{cases} 
2 & \text{(if the region } r_j \text{ touches the crossing } v_i \text{ twice)} \\
1 & \text{(if the region } r_j \text{ touches the crossing } v_i \text{ once)} \\
0 & \text{(otherwise)}
\end{cases}
\]
Theorem (Ahara-S.).

\[ A(\bar{D}) \in M_{n,n+2}(\mathbb{Z}) : \text{region choice matrix of a knot projection } \bar{D} \]
\[ b \in \mathbb{Z}^n : \text{a given integral vector (original scores of crossings)} \]
\[ \implies \text{There exists a solution } u \in \mathbb{Z}^{n+2} \text{ such that} \]
\[ A(\bar{D}) u + b = 0. \]

The region choice problem of the double counting rule is solvable.

Example

\[
\begin{pmatrix}
2 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{pmatrix}
= 
\begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{pmatrix} = 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]
Example

\[
\begin{pmatrix}
2 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

\[
u =
\begin{pmatrix}
x \\
y \\
-b_1 - 2x - y \\
2b_1 - b_2 - b_3 + b_4 + 3x + 2y \\
-b_1 + b_3 - b_4 - 2x - y \\
-b_1 + b_2 - b_4 - 2x - y
\end{pmatrix}
\]