Automorphisms of braid groups on orientable surfaces

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KIAS

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Let $\Sigma_g$ be a closed orientable surface of genus $g$.

- $\mathbf{p} = \{p_1, \ldots, p_p\} \subset \Sigma_g$: set of punctures, $\Sigma_{g,p} = \Sigma_g \setminus \mathbf{p}$
- $\mathbf{z} = \{z^0_1, \ldots, z^0_n\} \subset \Sigma_{g,p}$: set of marked points.

Figure: A surface $\Sigma_{g,p}$
ordered configuration space $F_n(\Sigma)$

$$F_n(\Sigma) = \{(z_1, \ldots, z_n) \in \Sigma^n | z_i \neq z_j \text{ if } i \neq j\},$$

unordered configuration space $B_n(\Sigma)$

$$B_n(\Sigma) = \{\{z_1, \ldots, z_n\} \subset \Sigma | z_i \neq z_j \text{ if } i \neq j\} = F_n(\Sigma)/S_n.$$

$n$-braid group $B_n(M) = \pi_1(B_n(M))$.

pure $n$-braid group $P_n(M) = \pi_1(F_n(M))$. 
What are $\text{Aut}(P_n(\Sigma_{g,p}))$ and $\text{Aut}(B_n(\Sigma_{g,p}))$?
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Let $\kappa := 2g + p + n$. 
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Let $\kappa := 2g + p + n$.

When $\kappa \leq 3$,

<table>
<thead>
<tr>
<th>$(g, p, n)$</th>
<th>$(0,0,1)$</th>
<th>$(0,0,2)$</th>
<th>$(0,0,3)$</th>
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<th>$(0,2,1)$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$B_n(\Sigma)$</td>
<td>${e}$</td>
<td>$\mathbb{Z}_2$</td>
<td>Dic$_{12}$</td>
<td>${e}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}^2$</td>
</tr>
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<td>Aut($B_n(\Sigma)$)</td>
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- Dic$_{12}$ : Dicyclic group of order 12.
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- Dic\(_{12}\) : Dicyclic group of order 12.

We assume that \( \kappa \geq 4 \).
extended Mapping class group

\[ \mathcal{M}_n^\pm(\Sigma_g, p) = \{ f \in \text{Homeo}^\pm(\Sigma_g, p) | f(z) = z \} / \text{isotopy} \].

extended Pure Mapping class group

\[ \mathcal{P}\mathcal{M}_n^\pm(\Sigma_g, p) = \{ f \in \text{Homeo}^\pm(\Sigma_g, p) | f|_z = \text{Id} \} / \text{isotopy} \].

Then

\[ \mathcal{P}\mathcal{M}_n^\pm(\Sigma_g, p) \subset \mathcal{M}_n^\pm(\Sigma_g, p) \subset \mathcal{M}_{p+n}(\Sigma_g) \],

and there is a commutative diagram with exact rows:

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathcal{P}n(\Sigma)/Z & \longrightarrow & \mathcal{P}\mathcal{M}_n^\pm(\Sigma) & \longrightarrow & \mathcal{M}_n^\pm(\Sigma) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \sim \\
1 & \longrightarrow & \mathcal{B}_n(\Sigma)/Z & \longrightarrow & \mathcal{M}_n^\pm(\Sigma) & \longrightarrow & \mathcal{M}_n^\pm(\Sigma) & \longrightarrow & 1
\end{array}
\]
Any \( f \in \mathcal{M}_n^\pm(\Sigma) \) induces an automorphism \( f_* \) on \( B_n(\Sigma) \). And so there exists a homomorphism \( \Psi : \mathcal{M}_n^\pm(\Sigma) \to \text{Aut}(B_n(\Sigma)) \).

Therefore,

\[
B_n(\Sigma)/Z \cong \text{Inn}(B_n(\Sigma)) \subset \mathcal{M}_n^\pm(\Sigma) \xrightarrow{\Psi} \text{Aut}(B_n(\Sigma)) \quad (2)
\]

**Lemma 1**

\( \Psi \) is injective.

By Eq. (2) and Lemma 1,

**Corollary 2**

\[
\mathcal{M}^\pm(\Sigma) \subset \text{Out}(B_n(\Sigma)).
\]
$f = \begin{array}{c}
\cdot \\
\cdot \\
\cdot
\end{array}$

\[ f_* \not\in \text{Inn}(B_n(\Sigma)) \]

$\beta = \in \Sigma \times I$

$g = \begin{array}{c}
\cdot \\
\circ \\
\cdot
\end{array}$

\[ g_* \in \text{Inn}(B_n(\Sigma)) \]

$= f_*(\beta)$

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Known Results — $n = 1$, $\Sigma$ is closed ($p = 0$)

Note that $B_1(\Sigma) = P_1(\Sigma) = \pi_1(\Sigma)$.

Theorem 3 (Dehn-Nielsen 1927)

If $\Sigma$ is closed ($p = 0$) and $g \geq 1$, then

$$\Aut(\pi_1(\Sigma)) \cong \mathcal{M}^\pm_1(\Sigma).$$
Known Results — $n = 1$, $\Sigma$ is not closed ($p \geq 1$)

Then

$$\pi_1(\Sigma) = \left\langle a_1, b_1, \ldots, a_g, b_g, \zeta_1, \ldots, \zeta_p \left| \prod_{r=1}^{g} [b_r^{-1}, a_r] \prod_{t=1}^{p} \zeta_t = e \right. \right\rangle$$

$$= \text{free group of rank } 2g + p - 1.$$
Theorem 4 (Nielsen 1924)

$\text{Aut}(F_m)$ is finitely presented for any $m < \infty$.

Note that $M_{1+}^\pm(\Sigma_{g,p}) \not\subseteq \text{Aut}(F_{2g+p-1})$.

Theorem 5 (Magnus 1934, Zieschang-Vogt-Coldewey 1970)

Suppose $p \geq 1$. Then $\phi \in \text{Aut}(\pi_1(\Sigma))$ is induced by a homeomorphism if and only if $\phi$ preserves the peripheral structure.

A peripheral structure $K$ of $\pi_1(\Sigma)$ is a set of all conjugates of $\zeta_t$'s.

$$K = \{ w^{-1}\zeta_t w | w \in \pi_1(\Sigma), 1 \leq t \leq p \}.$$
Known Results — Classical braid case ($\Sigma = \mathbb{R}^2$)

Theorem 6 (Dyer-Grossman 1981)

For $n \geq 2$,

$$\text{Aut}(B_n(\mathbb{R}^2)) \cong \mathcal{M}_n^\pm(\mathbb{R}^2).$$
Center $Z = Z(B_n(\Sigma)) = Z(P_n(\Sigma))$

1. $P_n(\Sigma) \simeq P_n(\Sigma)/Z \times Z$

2. There is a split short exact sequence

$$1 \longrightarrow tv(P_n(\Sigma)) \longrightarrow Aut(P_n(\Sigma)) \overset{\Phi}{\longrightarrow} Aut(P_n(\Sigma)/Z) \longrightarrow 1$$

where $tv(G) := \text{Ker}(Aut(G) \rightarrow Aut(G/Z(G)))$ is a transvection subgroup.

Observation

$$Aut(P_n(\Sigma)) \simeq tv(P_n(\Sigma)) \rtimes Aut(P_n(\Sigma)/Z).$$
Let $N = \left(\frac{n+p-1}{2}\right) - 1$. The transvection subgroups are as follows.

<table>
<thead>
<tr>
<th>$\Sigma$</th>
<th>$\text{tv}(\mathcal{P}_n(\Sigma))$</th>
<th>$\text{tv}(\mathcal{B}_n(\Sigma))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^2$, $n \geq 4$</td>
<td>$\mathbb{Z}_2^N$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\mathbb{R}^2$, $n \geq 3$</td>
<td>$\mathbb{Z}^N$</td>
<td>${e}$</td>
</tr>
<tr>
<td>$A$, $n \geq 2$</td>
<td>$\mathbb{Z}^N$</td>
<td>$\begin{cases} \mathbb{Z} &amp; n \geq 3 \ \mathbb{Z} \rtimes \mathbb{Z}_2 &amp; n = 2 \end{cases}$</td>
</tr>
<tr>
<td>$T$, $n \geq 2$</td>
<td>$\mathbb{Z}^{4(n-1)} \rtimes \text{GL}(2, \mathbb{Z})$</td>
<td>$\text{GL}(2, \mathbb{Z})[n]$</td>
</tr>
</tbody>
</table>

**Table:** Transvection subgroups

- $\text{GL}(2, \mathbb{Z})[n] := \text{Ker}(\text{GL}(2, \mathbb{Z}) \to \text{GL}(2, \mathbb{Z}_n))$. 
Theorem 8 (Ivanov 1992, Bellingeri 2008)

Let $n \neq 2$ and $\Sigma$ be any surface. If $n = 4$, assume furthermore that $\Sigma$ does not embed in $\mathbb{R}^2$. Then $P_n(\Sigma)$ is a characteristic subgroup of $B_n(\Sigma)$.

Theorem 8 holds for $\Sigma = T$ and $n = 2$. It is unknown when

1. $g = 0$, $p \geq 2$, $n \geq 2$;
2. $g \geq 2$, $n = 2$. 

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Suppose \( P_n(\Sigma) \) is a characteristic subgroup of \( B_n(\Sigma) \). Then there is a commutative diagram as follows.

\[
\begin{array}{cccccc}
1 & \longrightarrow & tv(B_n(\Sigma)) & \longrightarrow & Aut(B_n(\Sigma)) & \longrightarrow & Aut(B_n(\Sigma)/Z) \\
& & \downarrow \Gamma & & \downarrow \Gamma & & \downarrow \tilde{\Gamma} \\
1 & \longrightarrow & tv(P_n(\Sigma)) & \longrightarrow & Aut(P_n(\Sigma)) & \longrightarrow & Aut(P_n(\Sigma)/Z) & \longrightarrow & 1.
\end{array}
\]

In general, \( \Phi' \) is not surjective. (e.g., \( B_4(S^2) \)).

**Lemma 9**

All vertical maps are injective, and therefore

\[
Aut(B_n(\Sigma)/Z) \cong \frac{N_{Aut(P_n(\Sigma)/Z)}(B_n(\Sigma)/Z)}{C_{Aut(P_n(\Sigma)/Z)}(B_n(\Sigma)/Z)}.
\]
In summary, under the assumption that $P_n(\Sigma)$ is a characteristic subgroup, it is essential to compute

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$$\text{Aut}(P_n(\Sigma)/\mathbb{Z}).$$

Note that $P_n(\Sigma)/\mathbb{Z} = F_2$ when $\kappa = 4$.

**$\kappa = 4$ case**

$$\text{Aut}(P_n(\Sigma)) \cong \text{tv}(P_n(\Sigma)) \rtimes \text{Aut}(F_2).$$

Let $\Sigma = \Sigma_{g,p}$. Suppose $\kappa \geq 5$.

1. If $\chi(\Sigma) \geq -1$, then

$$\text{Aut}(P_n(\Sigma)) \simeq \text{tv}(P_n(\Sigma)) \rtimes M_{p+n}(\Sigma_g).$$

Moreover, if $g = 0$, then

$$\text{Aut}(B_n(\Sigma_{0,p})) \simeq \text{tv}(B_n(\Sigma_{0,p})) \rtimes M_{p+n}(\Sigma_{0,p}).$$

2. If $\Sigma = \Sigma_g$ is closed and $\chi(\Sigma) \leq -2$, then

$$\text{Aut}(P_n(\Sigma)) \simeq \text{Aut}(B_n(\Sigma)) \simeq M_n^{\pm}(\Sigma).$$
When $\chi(\Sigma) \leq -2$ and $\Sigma$ is not closed, then both $\text{Aut}(B_n(\Sigma))$ and $\text{Aut}(P_n(\Sigma))$ are unknown for all $n \geq 2$. 
Main Results I

**Theorem 11**

Let \( \Sigma = \Sigma_{g,p} \). Then \( P_n(\Sigma) \) is a characteristic subgroup of \( B_n(\Sigma) \) if and only if \( (g, p, n) \neq (0, 2, 2) \).

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**Figure:** \( B_2(A) = \langle \sigma_1, \zeta_1 | [\zeta_1, \sigma_1 \zeta_1 \sigma_1] \rangle \)

Let \( \sqrt{\pi} \in \text{Aut}(B_2(A)) \) defined by

\[
\sqrt{\pi}(\sigma_1) = \zeta_1^{-1}, \quad \sqrt{\pi}(\zeta_1) = \zeta_1 \sigma_1 \zeta_1.
\]

Then \( \sqrt{\pi}(P_2(A)) \not\subset P_2(A) \).
### Theorem 12

Suppose $\chi(\Sigma) \leq -2$ and $n \geq 2$. Then

$$\text{Aut}(P_n(\Sigma)) \simeq \mathcal{M}_n^\pm(\Sigma).$$

### Theorem 13

Suppose $n \geq 2$ and $(g, p, n) \neq (0, 2, 2)$. Then

$$\text{Aut}(B_n(\Sigma)) \simeq \text{tv}(B_n(\Sigma)) \rtimes \frac{\mathcal{M}_n^\pm(\Sigma)}{Z(\mathcal{M}_n^\pm(\Sigma))}.$$
Theorem 14

\[ \text{Aut}(B_2(A)) \cong \langle \sqrt{\pi} \rangle \rtimes \left( \text{tv}(B_2(A)) \rtimes \frac{\mathcal{M}_2^\pm(A)}{\mathbb{Z}(\mathcal{M}_2^\pm(A))} \right) / \langle \sqrt{\pi^2} = \pi \rangle \]

- \( \text{tv}(B_2(A)) = \langle \pi \rangle \rtimes \langle s|s^2 \rangle \), where \( s \) is the hyperelliptic involution.
Idea of the proof I

Let \( \Sigma_i = \Sigma \setminus z \cup \{z_i^0\} \). Then \( U_i := \pi_1(\Sigma_i, z_i^0) \) is a free, normal subgroup of \( P_n(\Sigma) \).

**Lemma 15**

Let \( \phi \in \text{Aut}(P_n(\Sigma)) \). Suppose that \( \phi(U_i) = U_i \). Then \( \phi|_{U_i} = Id \) if and only if \( \phi = Id \).
Idea of the proof II

Let $\phi \in \text{Aut}(P_n(\Sigma))$. We will show that there exists $f \in \mathcal{M}^\pm(\Sigma\hat{i})$ such that

1. $f_* \circ \phi(U_i) = U_i$ for all $i$.
2. $f_* \circ \phi$ preserves the peripheral structure of $U_i$.

**Theorem 5 (Magnus 1934, Zieschang-Vogt-Coldewey 1970)**

$f_* \circ \phi \in \text{Aut}(U_i)$ is induced by a homeomorphism $g \in \mathcal{M}^\pm(\Sigma\hat{i})$ if and only if $\phi$ preserves the peripheral structure of $U_i$.

Then by Theorem 5 and Lemma 15,

$$\phi = (f^{-1} \circ g)_* \in \text{Aut}(P_n(\Sigma))$$

where $f^{-1} \circ g \in \mathcal{M}_n^\pm(\Sigma)$. 
Let $X = \{ A_{i,j}, a_{i,r}, b_{i,r}, \zeta_{i,t} \mid \forall i, j, r, t \}$ be the union of all generators for $U_i$, and
\[
\tilde{X} = \{ w^{-1}xw \mid w \in P_n(\Sigma), x \in X \}.
\]

For any $\beta \in P_n(\Sigma)$,

1. $C(\beta)$ = centralizer of $\beta$
2. $Z(\beta) = Z(C(\beta))$ = center of $C(\beta)$ : free abelian group
3. $rk(\beta)$ = rank of $Z(\beta)$

**Lemma 16 (Irmak-Ivanov-McCarthy 2003, An)**

Let $\beta \in P_n(\Sigma)$. Then $rk(\beta) = 1$ and $C(\beta)/Z(\beta)$ is directly indecomposable if and only if $\beta \in \tilde{X} \cup \tilde{X}^{-1}$.

**Proposition 17**

\[
\phi(\tilde{X} \cup \tilde{X}^{-1}) = \tilde{X} \cup \tilde{X}^{-1}
\]
For $x \in \bar{X}$, $\text{end}(x)$ is either

\[ x = w^{-1}A_{i,j}w \quad \text{(a)} \quad \text{end}(x) = \{z_0^i, z_0^j\} \]

\[ x = w^{-1}\xi_{i,t}w \quad \text{(b)} \quad \text{end}(x) = \{z_0^i, p_t\} \]

\[ x = w^{-1}a_{i,r}w \quad \text{(c)} \quad \text{end}(x) = \{z_0^i\} \]

**Lemma 18**

Let $x, y \in \bar{X} \cup \bar{X}^{-1}$. Suppose that $|\text{end}(x)| = 2$ and $\text{end}(x) \cap \text{end}(y) = \emptyset$. Then there exists $w \in P_n(\Sigma)$ such that $[w^{-1}xw, y] = e$. 
Proposition 19

There exists a permutation $\rho(\phi)$ on $\{1, \ldots, n\}$ such that

$$\phi(U_i) = U_j$$

for $1 \leq i \leq n$ and $j = \rho(\phi)(i)$.

By this proposition, there exists $f \in M^\pm(\Sigma_{\tilde{i}})$ such that

$$f_\ast \circ \phi(U_i) = U_i$$

for all $i$. 
Proposition 20

$f_* \circ \phi$ preserves end.

Idea of the proof

By Lemma 18,

$$|\text{end}(x)| = |\text{end}(f_* \circ \phi(x))|.$$  

And consider a homomorphism $\mathbb{P}_n(\Sigma_{g,p}) \to \mathbb{P}_n(\Sigma_g)$ induced by inclusion $\Sigma_{g,p} \to \Sigma_g$. 

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**Proposition 21**

\( f_* \circ \phi \) preserves the peripheral structure.

**Idea of the proof**

It suffices to prove that \( f_* \circ \phi(\overline{X}) \) is either \( \overline{X} \) or \( \overline{X}^{-1} \).

The induced map \( \overline{f_* \circ \phi} \) on

\[
H_1(U_i) \cong \mathbb{Z}^{2g+p+n-2}
\]

is an isomorphism if and only if \( \overline{f_* \circ \phi} \) on \([X]\) is either \( I \) or \( -I \).