The purposes

Recently Dimca-Papadima [1] and Randell [2] proved that the complement $M(\mathcal{A})$ of a complex hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$ is homotopy equivalent to a minimal CW-complex, namely, a CW-complex whose number of $k$-cells is equal to its $k$-th betti number for each $k \geq 0$.

The goal here is to provide

- a presentation for the fundamental group $\pi_1(M(\mathcal{A}))$,
- a minimal chain complex computing local system homology obtained from the study of attaching maps for a real arrangement $\mathcal{A}$.

We concentrate on 2-dimensional case. See [3] for details.

Counting chambers

Let

$$F^0 \subset F^1 \subset F^2 = V_{\mathbb{R}} = \mathbb{R}^2$$

be a generic flag. And define

$$\text{ch}_k(\mathcal{A}) = \left\{ C \left| \begin{array}{l} C \cap F^{k-1} = \emptyset \\ C \cap F^k \neq \emptyset \end{array} \right. \right\},$$

where $C$ is a chamber.

**Fact:** $\#\text{ch}_k(\mathcal{A}) = b_k(M(\mathcal{A}))$.

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\[ \pi(\mathcal{A}, t) = 1 + 3t + 3t^2 \]
The minimal CW-decomposition

\[ \text{ch}_k(A) \] indexes \( k \)-cells.

\( k = 1 \).

\[ C_1 \] \[ C_2 \] \[ C_3 \]

F^0

F^1_C

\[ \gamma_1 \] \[ \gamma_2 \] \[ \gamma_3 \]

Cahmber: \( C_i \leftrightarrow 1 \)-cell: \( \gamma_i \).

\( k = 2 \). Let \( Q \) be a defining equation of \( A \). Let \( f \) be a defining equation of the line \( F^1 \). Then

\[ \varphi = \left| \frac{f^\lambda}{Q} \right| : M(A) \to \mathbb{R}_{\geq 0} \]

is a Morse function. \( C \in \text{ch}_2(A) \) is a stable manifold. The corresponding unstable manifold \( \sigma_C \) is the 2-cell attaching to \( M(A) \cap F^1_C \).

\[ \deg(C, C') = \begin{cases} 1 & \text{if } C = C_1, C'' = C_4 \\ 0 & \text{if } C = C_2, C'' = C_3 \\ -1 & \text{if } C = C_4, C'' = C_1 \end{cases} \]

The degree map will be used to describe the fundamental group and the boundary map of twisted minimal chain complex.
Fundamental group

\[ \text{ch}_1(\mathcal{A}) = \{C_1, \ldots, C_n\}. \]

Let \( \gamma_1, \ldots, \gamma_n \) be the corresponding 1-cells.

Given \( C \in \text{ch}_2(\mathcal{A}) \), define a word \( R(C) \) by

\[ R(C) := \gamma_1^{e_1} \gamma_2^{e_2} \cdots \gamma_n^{e_n} \gamma_1^{-e_1} \cdots \gamma_n^{-e_n}, \]

where \( e_i = \deg(C, C_i) \).

Theorem

The fundamental group \( \pi_1(M(\mathcal{A})) \) has the following presentation:

\[ \langle \gamma_1, \ldots, \gamma_n \mid R(C); C \in \text{ch}_2(\mathcal{A}) \rangle. \]

Local system

Let \( \mathcal{A} = \{H_1, \ldots, H_n\} \). Choose a nonzero complex number \( q_i \in \mathbb{C}^* \) for each \( i = 1, \ldots, n \).

We consider the rank one local system \( \mathcal{L} \) on \( M(\mathcal{A}) \) such that the local monodromy around \( H_i \) is \( q_i^2 \).

Definition

For two chambers \( C \) and \( C' \),

\[ \text{Sep}(C, C') := \{i | H_i \text{ separates } C, C'\}. \]

And define

\[ \Delta(C, C') := \prod_{i \in \text{Sep}(C, C')} q_i - \prod_{i \in \text{Sep}} q_i^{-1} \]

\[ \pi_1(M(\mathcal{A})) = \mathbb{Z}^3 \]
Twisted minimal chain complex

\[ \mathcal{A} = \{H_1, \ldots, H_n\}, \]
\[ \text{ch}_0(\mathcal{A}) = \{p\}, \]
\[ \text{ch}_1(\mathcal{A}) = \{\gamma_1, \ldots, \gamma_n\} \]
as above. Put

\[ C_0 := \mathbb{C}[p] \]
\[ C_1 := \mathbb{C}[\gamma_1] \oplus \cdots \oplus \mathbb{C}[\gamma_n] \]
\[ C_2 := \bigoplus_{C \in \text{ch}_2} \mathbb{C}[\sigma_C] \]
and define \( \partial : C_k \to C_{k-1} \) as

\[ \partial([\gamma_i]) = \Delta(\gamma_i, p)[p] \]
\[ \partial([\sigma_C]) = \sum_{i=1}^{n} \deg(C, \gamma_i) \Delta(C, \gamma_i)[\gamma_i]. \]

**Theorem**

Then \((C_\bullet, \partial)\) is a chain complex, namely, \( \partial^2 = 0 \) and

\[ H_k(C_\bullet, \partial) \cong H_k(M(\mathcal{A}), \mathcal{L}). \]

**Reference**

