LOW–DIMENSIONAL LINEAR REPRESENTATIONS OF MAPPING CLASS GROUPS.

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Let S denote a compact connected orientable surface of genus g and let $\mathrm{Mod}(S)$ denote the mapping class group of it, the group of isotopy classes of diffeomorphisms $S \to S$ which are identity on the boundary of S. If $\bar{S}$ is the closed surface obtained by gluing a disk along each boundary component, after fixing a symplectic basis of $H_1(\bar{S};\mathbb{Z})$, the action of $\mathrm{Mod}(\bar{S})$ on the first homology group $H_1(\bar{S};\mathbb{Z})$ gives rise to a homomorphism $\mathrm{Mod}(\bar{S}) \to \mathrm{Sp}(2g,\mathbb{Z})$ onto the symplectic group. Thus, one gets a homomorphism $P : \mathrm{Mod}(S) \to \mathrm{GL}(n,\mathbb{C})$ which is the composition of the following homomorphism

$$\mathrm{Mod}(S) \to \mathrm{Mod}(\bar{S}) \to \mathrm{Sp}(2g,\mathbb{Z}) \hookrightarrow \mathrm{GL}(2g,\mathbb{C}).$$

In my lectures, I am planning to give the detailed proof of the following three theorems:

Theorem 1. (Franks-Handel, Korkmaz) *Let $g \geq 1$ and let $n \leq 2g-1$. Let $\phi : \mathrm{Mod}(S) \to \mathrm{GL}(n,\mathbb{C})$ be a homomorphism. Then ϕ factors through $\mathrm{Mod}(S) \to H_1(\mathrm{Mod}(S);\mathbb{Z})$. In particular, the image of ϕ is*

(i) *trivial if $g \geq 3$, and*
(ii) *a quotient of the cyclic group $\mathbb{Z}_{10}$ of order 10 if $g = 2$.*

Theorem 2. (Korkmaz) *Let $g \geq 3$ and let $\phi : \mathrm{Mod}(S) \to \mathrm{GL}(2g,\mathbb{C})$ be a group homomorphism. Then ϕ is either trivial or conjugate to the homomorphism $P : \mathrm{Mod}(S) \to \mathrm{GL}(2g,\mathbb{C})$.*

Theorem 3. (Korkmaz) *Let $g \geq 3$ and let $n \leq 3g-3$. Then there is no injective homomorphism $\mathrm{Mod}(S) \to \mathrm{GL}(n,\mathbb{C})$.*

One of the outstanding unsolved problems in the theory of mapping class groups is the existence of a faithful linear representation $\mathrm{Mod}(S) \to \mathrm{GL}(n,\mathbb{C})$ for some n. It is known that the braid groups, the mapping class group of the sphere with marked points and the hyperelliptic mapping class groups are linear. The third theorem shows that in dimensions $n \leq 3g-3$, there is no faithful linear representation of the mapping class group. Previously, this was known for $n \leq \sqrt{g+1}$.

I will also discuss a few applications of these theorems, including some algebraic consequences.

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